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REDUCTIVE WEAK DECOMPOSABLE OPERATORS ARE SPECTRAL

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ABSTRACT. We show that if a bounded linear operator T on a complex Hilbert space is reductive and weak decomposable, then T is a spectral operator with a normal scalar part. This is a generalization of a result due to Jafarian [3].

1. Preliminaries. Let H be a complex Hilbert space. An operator T means a bounded linear transformation on H. For an operator T, $\sigma(T)$ denotes its spectrum and $\rho(T)$ denotes its resolvent set. An invariant subspace Y of T is called a spectral maximal space of T if $Z \subset Y$ for any invariant subspace Z of T such that $\sigma(T \mid Z) \subset \sigma(T \mid Y)$. SM$(T)$ denotes the family of all spectral maximal spaces of T. An operator T is called weak decomposable if for any open covering $\{G_1, \ldots, G_n\}$ of $\sigma(T)$ there exists a system $\{Y_1, \ldots, Y_n\}$ in SM(T) such that (1) $H = \overline{Y_1 + \cdots + Y_n}$ and (2) $\sigma(T \mid Y_i) \subset G_i$ for every $i = 1, \ldots, n$.

An operator T is said to have the property (A) or the single valued extension property if there exists no nonzero analytic function $f(z)$ such that $(z - T)f(z) \equiv 0$. If an operator T has (A), then for any $x \in H$ there exists a maximal analytic extension $f_x(z)$ of $(z - T)^{-1}x$ such that $(z - T)f_x(z) \equiv x$. $\rho_T(x)$ denotes the domain of $f_x(z)$ and $\sigma_T(x) = \mathbf{C} \setminus \rho_T(x)$. Let $H_T(E) = \{x \in H: \sigma_T(x) \subset E\}$ for any subset $E \subset \mathbf{C}$. An operator T with (A) is said to satisfy the condition (C) if $H_T(F)$ is closed for all closed subsets $F \subset \mathbf{C}$.

An operator T is called reductive if every invariant subspace of T reduces T.

2. Main results.

LEMMA 1. *For an operator T with* (A) *and for a subset $E \subset \mathbf{C}$, the following assertions are equivalent.*

(1) $\sigma(T \mid \overline{H_T(E)}) \subset E$.

(2) *$H_T(E)$ is closed.*

And in this case $H_T(E) \in$ SM(T).

PROOF. We show the implication (1) $\to$ (2). Let $x \in \overline{H_T(E)}$ be given, then $f(z) = (z - T \mid \overline{H_T(E)})^{-1}x$ is analytic and $(z - T)f(z) \equiv x$ on $\rho(T \mid \overline{H_T(E)}) \supset \mathbf{C} \setminus E$. Hence $\sigma_T(x) \subset E$ and $x \in H_T(E)$. Thus $H_T(E)$ is closed. The implication (2) $\to$ (1) and $H_T(E) \in$ SM(T) is well known by [**1**, Proposition 1.3.8].

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LEMMA 2. *If an operator T is reductive and weak decomposable, then T satisfies* (C).

PROOF. Since every weak decomposable operator T has (A) by [**2**, Theorem 2.3], we have only to show $\sigma(T|\overline{H_T(F)}) \subset F$ for all closed subsets $F \subset \mathbf{C}$ by Lemma 1. Let $\lambda_0 \notin F$ be given, then there exist an $\varepsilon > 0$ and $G_1 = \{\lambda: |\lambda - \lambda_0| < 2\varepsilon\}$ such that $G_1 \cap F = \phi$. Let $G_2 = \{\lambda: |\lambda - \lambda_0| > \varepsilon\}$, then $\{G_1, G_2\}$ is an open covering of $\sigma(T)$, hence there exist $Y_1, Y_2 \in \mathrm{SM}(T)$ such that $H = \overline{Y_1 + Y_2}$ and $\sigma(T| Y_i) \subset G_i$ for $i = 1, 2$. Let P denote the orthogonal projection of H onto $\overline{H_T(F)}$. Then $TP = PT$ by assumption and $\overline{H_T(F)} = PH = P(\overline{Y_1 + Y_2}) \subset \overline{PY_1 + PY_2}$. We show $PY_1 = \{0\}$. Since $Y_1 \in \mathrm{SM}(T)$ is hyperinvariant under T by [**1**, Proposition 1.3.2], we have $PY_1 \subset Y_1$ and $PY_1 \subset Y_1 \cap \overline{H_T(F)}$. Let P_1 denote the orthogonal projection of H onto Y_1. Then $TP_1 = P_1T$, hence for any $x \in H_T(F)$ we have $\sigma_T(P_1x) \subset \sigma_T(x) \cap \sigma(T| Y_1) \subset F \cap G_1 = \phi$. Hence $P_1x = 0$ and it follows immediately that $Y_1 \perp \overline{H_T(F)}$. Thus $PY_1 \subset Y_1 \cap \overline{H_T(F)} = \{0\}$ and $PY_1 = \{0\}$. Hence $\overline{H_T(F)} \subset \overline{PY_1 + PY_2} \subset \overline{PY_2} \subset Y_2$ because $PY_2 \subset Y_2$. Since $\overline{H_T(F)}$ is a reducing subspace of T, we have

$$\sigma_{T|\overline{H_T(F)}}(x) = \sigma_T(x) \subset \sigma(T| Y_2) \subset G_2$$

for all $x \in \overline{H_T(F)}$. Hence

$$\sigma\left(T| \overline{H_T(F)}\right) = \bigcup \left\{\sigma_{T|\overline{H_T(F)}}(x): x \in \overline{H_T(F)}\right\} \subset G_2,$$

and thus $\lambda_0 \neq \sigma(T|\overline{H_T(F)})$.

THEOREM. *If an operator T is reductive and weak decomposable, then $T = N + Q$ where N is a normal operator and Q is a quasinilpotent operator commuting with N, i.e. T is a spectral operator with a normal scalar part.*

PROOF. We show first $\sigma(T|H_T(F)^\perp) \subset \overline{\mathbf{C}\setminus F}$ for all closed subsets $F \subset \mathbf{C}$. Let $\lambda_0 \notin \overline{\mathbf{C}\setminus F}$ be given, then there exists an $\varepsilon > 0$ and $G_1 = \{\lambda: |\lambda - \lambda_0| < 2\varepsilon\}$ such that $G_1 \cap \overline{\mathbf{C}\setminus F} = \varnothing$. Let $G_2 = \{\lambda: |\lambda - \lambda_0| > \varepsilon\}$, then $\{G_1, G_2\}$ is an open covering of $\sigma(T)$, hence there exist $Y_1, Y_2 \in \mathrm{SM}(T)$ such that $H = \overline{Y_1 + Y_2}$ and $\sigma(T| Y_i) \subset G_i$ for $i = 1, 2$. Let P denote the orthogonal projection of H onto $H_T(F)^\perp$. Then $TP = PT$ by assumption and $H_T(F)^\perp = PH = P(\overline{Y_1 + Y_2}) \subset \overline{PY_1 + PY_2}$. We show $PY_1 = \{0\}$. Let $x \in Y_1$ be given, then $\sigma_T(Px) \subset \sigma_T(x) \subset \sigma(T| Y_1) \subset G_1 \subset F$. Hence $Px \in H_T(F) \cap H_T(F)^\perp = \{0\}$ and $PY_1 = \{0\}$. Hence $H_T(F)^\perp \subset \overline{PY_1 + PY_2} \subset \overline{PY_2} \subset Y_2$. Since $H_T(F)^\perp$ is a reducing subspace of T, we have $\sigma(T|H_T(F)^\perp) \subset \sigma(T| Y_2) \subset G_2$, and thus $\lambda_0 \notin \sigma(T|H_T(F)^\perp)$. Hence $\sigma(T|H_T(F)^\perp) \subset \overline{\mathbf{C}\setminus F}$ and $H_T(F)^\perp \subset H_T(\overline{\mathbf{C}\setminus F})$. Hence $H = H_T(F) + H_T(F)^\perp \subset H_T(F) + H_T(\overline{\mathbf{C}\setminus F})$ and so $H = H_T(F) + H_T(\overline{\mathbf{C}\setminus F})$ for all closed subsets $F \subset \mathbf{C}$. Wadhwa [**5**, Corollary 4] shows that if a reductive operator T has (A) and satisfies (C) and $H = H_T(F) + H_T(\overline{\mathbf{C}\setminus F})$ for all closed subsets $F \subset \mathbf{C}$, then T is a spectral operator with a normal scalar part. Hence the proof is completed by Lemma 2.

REMARK. We can relax the condition of weak decomposability by weak 2-decomposability.

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REFERENCES

1. I. Colojoară and C. Foiaş, *Theory of generalized spectral operators*, Gordon & Breach, New York, 1968.

2. A. A. Jafarian, *Weak and quasi-docomposable operators*, Rev. Roumaine Math. Pures Appl. **22** (1977), 195–212.

3. ______, *On reductive operators*, Indiana Univ. Math. J. **23** (1974), 607–613.

4. B. L. Wadhwa, *Decomposable and spectral operators on a Hilbert space*, Proc. Amer. Math. Soc. **40** (1973), 112–114.

5. ______, *A note on reductive operators*, Acta Sci. Math. (Szeged) **38** (1976), 187–189.

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