

A CHAOTIC FUNCTION WITH SOME EXTREMAL PROPERTIES

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ABSTRACT. For a continuous function chaotic in the sense of Li and Yorke the continuum hypothesis implies the existence of a scrambled set which has among others full outer Lebesgue measure.

A continuous function $f: I \rightarrow I$, where I is a real interval, is called chaotic provided there is an uncountable set S (a scrambled set) such that for each $x, y \in S$, $x \neq y$, and each periodic point p of f ,

$$(1) \quad \limsup_{n \rightarrow \infty} |f^n(x) - f^n(y)| > 0, \quad \limsup_{n \rightarrow \infty} |f^n(x) - f^n(p)| > 0$$

and

$$(2) \quad \liminf_{n \rightarrow \infty} |f^n(x) - f^n(y)| = 0$$

where f^n is the n th iterate of f (cf. [1]). In the literature there is known no set S with positive Lebesgue measure. On the other hand, it is easy to construct from a Cantor-type function a chaotic continuous function whose each scrambled set is a null set. In the present note we prove the following

THEOREM. Let $f: [0, 1] \rightarrow [0, 1]$ be defined by $f(x) = 2x$ for $x \leq \frac{1}{2}$, and $f(x) = 2 - 2x$ otherwise. Then the continuum hypothesis implies the existence of a scrambled set S for f with outer Lebesgue measure 1 and such that the inequalities (1) can be replaced by

$$(3) \quad \limsup_{n \rightarrow \infty} |f^n(x) - f^n(y)| = 1,$$

$$(4) \quad \limsup_{n \rightarrow \infty} |f^n(x) - f^n(p)| \geq \frac{1}{2}.$$

First we prove some lemmas. In the sequel, f is always the above-quoted function and μ is the Lebesgue measure.

LEMMA 1. Let $I \subset (0, 1)$ be an open interval and M an infinite set of positive integers. Then there is a G_δ set $B = B(I, M) \subset (0, 1)$ with $\mu(B) = 1$ such that for each $x \in B$ there is infinitely many $n \in M$ with $f^n(x) \in I$.

PROOF. Denote $f^{-n}(I) = \{x; f^n(x) \in I\}$. For positive integers $m < n$ let $C_{m,n} = \cup \{f^{-i}(I); i \in M, m < i < n\}$. First assume that there is a sequence $n(1) < n(2) < \dots$ of integers such that $\mu(C_{n(i), n(i+1)}) = 1$ for each i ; then clearly $B = \bigcap_{i=1}^{\infty} C_{n(i), n(i+1)}$ is the desired set.

Received by the editors February 4, 1982.

1980 *Mathematics Subject Classification.* Primary 26A18; Secondary 54H20.

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 0002-9939/82/0000-0661/\$01.50

Otherwise choose $n(1) \in M$ such that $\mu(C_{n(1),m}) < 1$ for each $m > n(1)$, and put $B_1 = f^{-n(1)}(I)$. Clearly $\mu(B_1) = \mu(I) = d$. Now assume by induction that B_i are defined such that

$$(5) \quad \text{each } B_i \text{ is open and } B_{i-1} \subset B_i \subsetneq (0, 1),$$

$$(6) \quad \mu(B_i \setminus B_{i-1}) > \frac{d}{2}(1 - \mu(B_{i-1})),$$

$$(7) \quad f^{n(i)}(x) \in I \text{ for each } x \in B_i \setminus B_{i-1},$$

where $n(i) \in M, i = 2, 3, \dots, k$. For $n > n(k)$ let G_n be the system of intervals $(2^{-n}j, 2^{-n}(j + 1)), j = 0, 1, 2, \dots$, which are contained in $(0, 1) \setminus B_k$, and let $D_n = \cup G_n$. Clearly $D_n \cap B_k = \emptyset$ and for some sufficiently large $n = n(k + 1) \in M$ we have $\mu(D_n) > \frac{1}{2}(1 - \mu(B_k))$. Put $B_{k+1} = B_k \cup (f^{-n(k+1)}(I) \cap D_{n(k+1)})$. Now it is easy to verify that (5) to (7) are satisfied for $i = k + 1$. Denote $\lim_{k \rightarrow \infty} \mu(B_k) = b$. If $b < 1$, then by (6) $\mu(B_{k+1} \setminus B_k) > d(1 - b)/2 > 0$ for each k , and consequently by (5), $\lim_{k \rightarrow \infty} \mu(B_k) = \infty$, a contradiction. Hence $b = 1$.

Denote $E_0 = \cup_{k=1}^{\infty} B_k$. Now if we repeat the construction starting from $n(1)$ satisfying the condition $n(1) > i$ we obtain an open set $E_i, i = 1, 2, \dots$, with $\mu(E_i) = 1$ and such that for each $x \in E_i$ there is some $n \in M, n > i$ such that $f^n(x) \in I$. It suffices to take $B = \cap_{i=0}^{\infty} E_i$ and the lemma is proved.

LEMMA 2. *There is a G_δ set $A \subset (0, 1)$ with $\mu(A) = 1$ such that for each $x \in A$,*

$$\liminf_{n \rightarrow \infty} f^{n!}(x) = 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} f^{n!}(x) = 1.$$

PROOF. Put $M = \{n!\}_{n=1}^{\infty}, A_k = B((0, 1/k), M)$ and $A^k = B((1 - 1/k, 1), M)$ (see Lemma 1). Then $A = \cap_{k=1}^{\infty} (A_k \cap A^k)$ has the desired properties.

LEMMA 3. *If $x \in A$ and p is a periodic point of f , then (4) is true.*

PROOF. Let k be the order of p . Then $f^{kn}(p) = p$ for each n , hence $\lim_{n \rightarrow \infty} f^{n!}(p) = p$ and (4) follows from Lemma 2.

LEMMA 4. *For each $x \in A$ there is a G_δ set $A(x) \subset A$ with $\mu(A(x)) = 1$ such that for any $y \in A(x)$, statements (2) and (3) are satisfied.*

PROOF. Let $M_k = \{m; f^m(x) < 1/k\}$. Put $B_k = B((0, 1/k), M_k)$ and $B^k = B((1 - 1/k, 1), M_k)$. Now let $A(x) = A \cap \cap_{k=1}^{\infty} (B_k \cap B^k)$; by Lemma 1, $A(x)$ has the desired properties.

PROOF OF THEOREM. We use transfinite induction to construct the set S . By the continuum hypothesis there exists a well-ordering $\{P_\alpha\}_{\alpha < \Omega}$ of the system of perfect subsets P of $(0, 1)$ with $\mu(P) > 0$, where Ω is the first uncountable ordinal. Let $x_0 \in A \cap P_0$. When $\{x_\alpha\}_{\alpha < \beta}$ are defined such that for each $x = x_\alpha, y = x_\gamma$, where $\alpha < \gamma < \beta$, the inequalities (2) and (3) are satisfied, take $H = \cap_{\alpha < \beta} A(x_\alpha)$ (see Lemma 4). Then $\mu(H) = 1$ since H is a countable intersection of sets of full measure. Hence we can choose some $x_\beta \in H \cap P_\beta$. Denote $S = \{x_\alpha\}_{\alpha < \Omega}$. Clearly (2) and (3) are satisfied for any $x, y \in S, x \neq y$. Since $S \subset A$, Lemma 3 implies that for any $x \in S$ and any periodic point p of f , it follows that (4) is true. And finally, since S intersects each perfect subset of $(0, 1)$ of positive measure, the set S must have the full outer Lebesgue measure. Q.E.D.

REMARK. The above quoted function f has no scrambled set with a positive Lebesgue measure.

To see it assume the contrary: Let A be a scrambled set for f with $\mu(A) = \lambda > 0$. Fix some integer m such that $2^m \lambda > 1$. Clearly f^m maps each of the intervals $I_i = (2^{-m}(i-1), 2^{-m}i)$ linearly onto $(0, 1)$, $i = 1, \dots, 2^m$. Put $A_i = A \cap I_i$. Then $\sum_i \mu(f^m(A_i)) > 1$ hence for some $i \neq j$ we have $f^m(A_i) \cap f^m(A_j) \neq \emptyset$. Consequently there are two different points $x \in A_i, y \in A_j$ with $f^n(x) = f^n(y)$ for every $n > m$, contrary to the first inequality in (1).

REFERENCES

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