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TRACIAL POSITIVE LINEAR MAPS OF C^*-ALGEBRAS

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ABSTRACT. A positive linear map $\Phi: \mathfrak{A} \to \mathfrak{B}$ between two C^*-algebras is said to be *tracial* if $\Phi(A_1A_2) = \Phi(A_2A_1)$ for all $A_i \in \mathfrak{A}$. A tracial positive linear map $\mathfrak{A} \to \mathfrak{B}(\mathcal{H})$ is analyzed as the composition of a tracial positive linear map $\mathfrak{A} \to C(X)$ followed by a positive linear map $C(X) \to \mathfrak{B}(\mathcal{H})$.

Tracial positive linear maps are the natural generalizations of tracial states on C^*-algebras. We invite special attention to the natural occurrence of tracial positive linear maps in the study of finite von Neumann algebras, Toeplitz operators, as well as others (see Examples 1–5 in the context).

In consideration of the general global structure, we are concerned with two familiar classes of tracial positive linear maps: The first is the class of tracial positive linear maps from a C^*-algebra $\mathfrak{A}$ into a commutative C^*-algebra $C(X)$—actually, each such map can be described as a *continuous* (with respect to the compact Hausdorff space X) family of finite traces on $\mathfrak{A}$. The second class consists of positive linear maps from a commutative C^*-algebra $C(X)$ into $\mathfrak{B}(\mathcal{H})$. The main theorem asserts that the compositions of these two classes exhaust all; namely, each tracial positive linear map $\mathfrak{A} \to \mathfrak{B}(\mathcal{H})$ admits a factorization $\mathfrak{A} \to C(X) \to \mathfrak{B}(\mathcal{H})$ through a commutative C^*-algebra $C(X)$. Therefore, every tracial positive linear map is completely positive, and consequently, each contractive tracial positive linear map $\Phi: \mathfrak{A} \to \mathfrak{B}$ satisfies the Schwarz inequality $\Phi(A^*A) \geq \Phi(A^*)\Phi(A)$. This answers a question raised in [**4**].

Throughout this paper, general C^*-algebras are written in the German type $\mathfrak{A}$, $\mathfrak{B}$. We denote by $\mathfrak{B}(\mathcal{H})$ (resp. $\mathcal{K}(\mathcal{H})$) for the C^*-algebra of all bounded operators (resp. all compact operators) on a Hilbert space $\mathcal{H}$. A linear map $\Phi: \mathfrak{A} \to \mathfrak{B}$ is said to be *tracial* if $\Phi(A_1A_2) = \Phi(A_2A_1)$ for all A_i in $\mathfrak{A}$. A linear map $\Phi: \mathfrak{A} \to \mathfrak{B}$ is said to be *positive* if $\Phi(A)$ is positive for every positive $A \in \mathfrak{A}$. For each operator A, we write $C^*(A)$ for the C^*-algebra generated by A.

We begin with several examples to illustrate the natural occurrence of tracial linear maps in structure theory.

EXAMPLE 1. *If* $\mathfrak{A}$ *is a unital C^*-algebra with a unique tracial state* τ (in particular, if $\mathfrak{A}$ is a finite factor), *then every tracial positive linear map* $\Phi: \mathfrak{A} \to \mathfrak{B}(\mathcal{H})$ *is of the form*

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$\Phi(A) = \tau(A)\Phi(I)$. Hence Φ is completely determined by a single positive operator $\Phi(I) \in \mathfrak{B}(\mathcal{H})$. To demonstrate this, we first assume $\Phi(I) = I$; then each norm-1 vector $\xi \in \mathcal{H}$ induces a tracial state $(\Phi(\cdot)\xi, \xi)$ on $\mathfrak{A}$ and

$$((\Phi(A) - \tau(A)I)\xi, \xi) = (\Phi(A)\xi, \xi) - \tau(A) = 0;$$

thus $\Phi(A) = \tau(A)I$. In general, Φ need not be unital, but we still have $\|\Phi\| I \geqslant \Phi(I)$. Define Ψ: $\mathfrak{A} \to \mathfrak{B}(\mathcal{H})$ by

$$\Psi(A) = [\Phi(A) + \tau(A)(\|\Phi\| I - \Phi(I))]/\|\Phi\|.$$

Then Ψ is a unital tracial positive linear map. From the argument above, we get $\Psi(A) = \tau(A)I$, and consequently, $\Phi(A) = \tau(A)\Phi(I)$ as desired.

EXAMPLE 2. Let $\mathfrak{R}$ be a *finite* von Neumann algebra and let $\mathfrak{Z}(\mathfrak{R})$ be the centre of $\mathfrak{R}$. By a result of Dixmier, there is a unique tracial positive linear map Φ: $\mathfrak{R} \to \mathfrak{Z}(\mathfrak{R})$ such that $\Phi(Z) = Z$ for all $Z \in \mathfrak{Z}(\mathfrak{R})$. What really plays the central role in the structure theory is Dixmier's Approximation Theorem: For each $A \in \mathfrak{R}$, there is a unique $T_A \in \mathfrak{Z}(\mathfrak{R})$ such that $T_A \in$ the norm closed convex hull of $\{U^*AU$: U runs through all unitary operators in $\mathfrak{R}\}$. Henceforth, the assignment $A \mapsto T_A$ defines a tracial expectation Φ: $\mathfrak{R} \to \mathfrak{Z}(\mathfrak{R})$. Indeed, the properties above also characterize the finiteness of von Neumann algebras (see [**6**, Chapter IV, §§5 and 8] for details).

EXAMPLE 3. Let $\mathfrak{R}$ be a *properly infinite* von Neumann algebra. Then *the only tracial positive linear map* Φ: $\mathfrak{R} \to \mathfrak{B}(\mathcal{H})$ *is the trivial map* $\Phi(A) = 0$ *for all* $A \in \mathfrak{R}$. To see this, note that there exist isometries $S_1, S_2 \in \mathfrak{R}$ such that $I \geqslant S_1S_1^* + S_2S_2^*$ [**6**, Corollary 2, p. 298]. Hence any tracial positive linear map Φ defined on $\mathfrak{R}$ must satisfy

$$\Phi(I) \geqslant \Phi(S_1S_1^*) + \Phi(S_2S_2^*) = \Phi(S_1^*S_1) + \Phi(S_2^*S_2) = 2\Phi(I).$$

Thus $\Phi(I) = 0$, and Φ is the trivial map.

EXAMPLE 4. Let $H = l^2$ and let $S \in \mathfrak{B}(\mathcal{H})$ be the unilateral shift operator. We will exhibit *a tracial positive linear map* Φ: $C^*(S) \to C^*(S)$ *such that* $\Phi(\Phi(A)) = \Phi(A)$ *and* $\Phi(A) - A$ *is a compact operator for each* $A \in C^*(S)$.

We recall that $T \in \mathfrak{B}(\mathcal{H})$ is a *Toeplitz operator* iff $T = S^*TS$ (see [**8**, Chapter 7] for all relevant information about Toeplitz operators). It is well known that for each $A \in C^*(S)$, there is a *unique* Toeplitz operator T_A such that $T_A - A$ is compact. By other structure theorems, the assignment $A \mapsto T_A$ actually defines a tracial positive linear map Φ: $C^*(S) \to C^*(S)$ with the prescribed properties.

Alternatively, it may be worthwhile to study Toeplitz operators on a Hardy space. Let $\mathbf{T}$ be the unit circle, let $\mathcal{H}$ be the Hardy space $H^2(\mathbf{T})$, and let P be the projection from $L^2(\mathbf{T})$ onto $\mathcal{H}$. There arises naturally a positive linear map Θ: $C(\mathbf{T}) \to \mathfrak{B}(\mathcal{H})$ sending continuous functions onto "Toeplitz operators with continuous symbols"; namely, each $\phi \in C(\mathbf{T})$ defines the multiplication operator $M_\phi \in \mathfrak{B}(L^2(\mathbf{T}))$ and thus *the Toeplitz operator with symbol* ϕ, $T_\phi = PM_\phi|_{\mathcal{H}} \in \mathfrak{B}(\mathcal{H})$. It is a familiar fact that the unilateral shift operator $S \in \mathfrak{B}(l^2)$ is unitarily equivalent to the Toeplitz operator with symbol z, T_z, where $z \in C(\mathbf{T})$ denotes the identity function $\phi(z) = z$. Thus $C^*(S)$ is identifiable with $C^*(T_z)$. Moreover, it is also well known that $C^*(T_z) = \mathcal{K}(\mathcal{H}) + \{$Toeplitz operators with continuous symbols$\}$ and there is a natural $*$-isomorphism $C(\mathbf{T}) \simeq C^*(T_z)/\mathcal{K}(\mathcal{H})$ assigning $z \in C(\mathbf{T})$ to $T_z + \mathcal{K}(\mathcal{H})$. Henceforth, the composition of natural maps

$$C^*(T_z) \to C^*(T_z)/\mathcal{K}(\mathcal{H}) \simeq C(\mathbf{T}) \xrightarrow{\Theta} \mathfrak{B}(\mathcal{H})$$

becomes a tracial positive linear map Φ: $C^*(T_z) \to \mathfrak{B}(\mathcal{H})$ sending $C^*(T_z)$ onto {all Toeplitz operators with continuous symbols} with $\Phi \circ \Phi = \Phi$, and $\Phi(A) - A \in \mathcal{K}(\mathcal{H})$ for each $A \in C^*(T_z)$.

EXAMPLE 5. We may generalize the result of Example 4 to the utmost as follows. Let $\mathfrak{A}$ be a separable C^*-algebra and let $\mathfrak{J}$ be a closed two-sided ideal of $\mathfrak{A}$. Then *$\mathfrak{A}/\mathfrak{J}$ is commutative iff there is a tracial positive linear map Φ: $\mathfrak{A} \to \mathfrak{A}$ such that $\Phi(A) - A \in \mathfrak{J}$ and $\Phi(\Phi(A)) = \Phi(A)$ for each $A \in \mathfrak{A}$*. To demonstrate this, we let Π: $\mathfrak{A} \to \mathfrak{A}/\mathfrak{J}$ be the natural quotient map. The "if" part follows immediately from the fact $\Pi \circ \Phi = \Pi$ and

$$\begin{aligned}\Pi(A_1)\Pi(A_2) &= \Pi(A_1A_2) = \Pi(\Phi(A_1A_2)) = \Pi(\Phi(A_2A_1)) \\ &= \Pi(A_2A_1) = \Pi(A_2)\Pi(A_1).\end{aligned}$$

Conversely, suppose $\mathfrak{A}/\mathfrak{J}$ is commutative, then any positive linear map Ψ: $\mathfrak{A}/\mathfrak{J} \to \mathfrak{A}$ making the diagram

$$\begin{tikzcd} & \mathfrak{A} \arrow[d, "\Pi"] \\ \mathfrak{A}/\mathfrak{J} \arrow[ur, dashed, "\Psi"] \arrow[r, equal] & \mathfrak{A}/\mathfrak{J} \end{tikzcd}$$

commutative (see e.g. [**12**, Theorem 14] for the existence of such lifting) will induce a tracial linear map $\Phi = \Psi \circ \Pi$ satisfying $\Phi \circ \Phi = \Phi$ and $\Phi(A) - A \in \mathfrak{J}$ for all $A \in \mathfrak{J}$.

Now, we proceed to establish the main result.

THEOREM. *Let Φ: $\mathfrak{A} \to \mathfrak{B}(\mathcal{H})$ be a tracial positive linear map. Then there exist a commutative C^*-algebra $C(X)$ and tracial positive linear maps Φ_1: $\mathfrak{A} \to C(X)$, Φ_2: $C(X) \to \mathfrak{B}(\mathcal{H})$ such that $\Phi = \Phi_2 \circ \Phi_1$. Moreover, in case Φ is unital, then we can also require Φ_1 and Φ_2 to be unital.*

PROOF. Consider the second dual map Φ^{**}: $\mathfrak{A}^{**} \to \mathfrak{B}(\mathcal{H})^{**}$ which is σ-weakly continuous and positive. Because multiplication is separately continuous in the σ-weak topology on $\mathfrak{A}^{**}$, the presumed equality $\Phi(A_1A_2) = \Phi(A_2A_1)$ (with A_1, $A_2 \in \mathfrak{A}$) persists for Φ^{**} (with A_1, $A_2 \in \mathfrak{A}^{**}$); thus Φ^{**} is tracial. To climb down from $\mathfrak{B}(\mathcal{H})^{**}$, we appeal to the fact that $\mathfrak{B}(\mathcal{H})^{**}$ is the enveloping von Neumann algebra for $\mathfrak{B}(\mathcal{H})$ (or we appeal to the "injectivity" of $\mathfrak{B}(\mathcal{H})$); hence there exists a $*$-homomorphism Π: $\mathfrak{B}(\mathcal{H})^{**} \to \mathfrak{B}(\mathcal{H})$ such that $\Pi|_{\mathfrak{B}(\mathcal{H})} =$ the identity map on $\mathfrak{B}(\mathcal{H})$ (see [7, §12.1.5, p. 266]). Therefore, we get a tracial positive linear map $\Psi = \Pi \circ \Phi^{**}$: $\mathfrak{A}^{**} \to \mathfrak{B}(\mathcal{H})$ satisfying

$$\Psi|_{\mathfrak{A}} = \Pi \circ \Phi^{**}|_{\mathfrak{A}} = \Pi \circ \Phi = \Phi.$$

Next, write $\mathfrak{A}^{**} = \mathfrak{R}_1 \oplus \mathfrak{R}_2$ where $\mathfrak{R}_1$ is a finite von Neumann algebra and $\mathfrak{R}_2$ is a properly infinite von Neumann algebra. As already shown in Example 3 above, $\Psi|_{\mathfrak{R}_2}$ is trivial; we may ignore $\mathfrak{R}_2$ completely. By Dixmier's Approximation Theorem (as mentioned in Example 2), there is a tracial positive linear map $\Theta\colon \mathfrak{R}_1 \to \mathfrak{Z}(\mathfrak{R}_1)$ assigning each $A \in \mathfrak{R}_1$ to the unique element in the intersection of $\mathfrak{Z}(\mathfrak{R}_1)$ and the norm closure of $\{\sum \lambda_j U_j^* A U_j\colon \lambda_j \geq 0,\ \sum \lambda_j = 1,\ U_j$ are unitary operators in $\mathfrak{R}_1\}$. Since

$$\Psi\left(\sum \lambda_j U_j^* A U_j\right) = \sum \lambda_j \Psi(U_j^* A U_j) = \sum \lambda_j \Psi(A) = \Psi(A),$$

we have $\Psi(\Theta(A)) = \Psi(A)$. Altogether, we get a commutative diagram

$$\begin{tikzcd}
\mathfrak{A}^{**} = \mathfrak{R}_1 \oplus \mathfrak{R}_2 \arrow[r, "\Pi_1"] \arrow[drr, "\Psi"] & \mathfrak{R}_1 \arrow[r, "\Theta"] & \mathfrak{Z}(\mathfrak{R}_1) \arrow[d, "\Psi|_{\mathfrak{Z}(\mathfrak{R}_1)}"] \\
\mathfrak{A} \arrow[u, hook] \arrow[rr, "\Phi"'] & & \mathfrak{B}(\mathcal{H})
\end{tikzcd}$$

where Π_1 is the natural projection map $A_1 \oplus A_2 \mapsto A_1$. Letting $\Phi_1 = \Theta \circ \Pi_1|_{\mathfrak{A}}$ and $\Phi_2 = \Psi|_{\mathfrak{Z}(\mathfrak{R}_1)}$, we get a tracial positive factorization $\Phi = \Phi_2 \circ \Phi_1$ as desired. ∎

As an easy consequence of Kadison's inequality, each unital positive linear map $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ also satisfies the inequality

$$\Phi(A^*A) + \Phi(AA^*) \geq \Phi(A^*)\Phi(A) + \Phi(A)\Phi(A^*)$$

for all $A \in \mathfrak{A}$ (see [**10**, Lemma 7.3]). It may be of interest to see that in case $\Phi(A^*A) = \Phi(AA^*)$ for all $A \in \mathfrak{A}$, we can really split the inequality as follows.

COROLLARY. *Let $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ be a contractive tracial positive linear map between two C^*-algebras. Then $\Phi(A^*A) \geq \Phi(A^*)\Phi(A)$ for all $A \in \mathfrak{A}$.*

PROOF. From the theorem above, it follows that Φ is completely positive. It is well known that each contractive completely positive linear map has the inequality as asserted. ∎

Finally, we pose a

QUESTION. Let $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ be a tracial positive linear map. Does Φ admit a factorization $\mathfrak{A} \xrightarrow{\Phi_1} C(X) \xrightarrow{\Phi_2} \mathfrak{B}$ where $C(X)$ is a commutative C^*-algebra and Φ_1, Φ_2 are tracial positive linear maps?

As already revealed in the proof of Theorem, the question above has an affirmative answer if $\mathfrak{A}$ or $\mathfrak{B}$ is a W^*-algebra, or if $\mathfrak{B}$ is an injective C^*-algebra.

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