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METRIC INVARIANCE OF HAAR MEASURE

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ABSTRACT. Let d be a left invariant metric for a locally compact group G. We prove that isometric subsets of (G, d) have equal Haar measure.

1. Introduction. In elementary geometry we learn that "congruent figures in plane have equal area". This concerns the Euclidean metric and it is not quite clear whether isometric sets with respect to other metrics on R^2 or R^n have equal Lebesgue measure. Fickett and Mycielski [3] gave an affirmative answer for a large class of metrics including all norms. This result is extended below to all translation-invariant metrics on R^n.

In fact we consider a more general situation. Let (X, d) be a metric space and μ a Borel measure on X. We say that μ is invariant with respect to d, or d-invariant, if isometric Borel sets of X have equal measure. D and F are isometric if there is a mapping f from D onto F with $d(f(x), f(y)) = d(x, y)$ for all x, y in D.

Ulam asked whether the Lebesgue product measure on Hilbert's cube is invariant with respect to certain familiar metrics. Although a number of positive results were obtained by Mycielski [**8, 9**] the question remains unresolved. We do not touch upon this problem here but confine our attention to Haar measure on locally compact groups [**6, 10, 4**] for which the question of metric invariance can be answered in a simple way.

THEOREM 1. *Let (G, d) be a metric, locally compact group, and let d be left invariant. Then any left Haar measure on G is d-invariant.*

Throughout the paper we shall assume that d is a left invariant metric for the locally compact group G, A is a compact subset of G with nonempty interior and λ is a Haar measure. For the proof of Theorem 1 it suffices to construct a d-invariant Borel measure μ on G with $0 < \mu(A) < \infty$. Since left translations are isometries, μ will be a Haar measure and hence every Haar measure on G will be d-invariant.

To find μ we combine the Weil-Cartan construction of Haar measure with Hausdorff's measure construction.

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2. Hausdorff measure. The h-dimensional Hausdorff measure μ^h on a metric space (X, d) is defined by

$$\mu^h(D) = \lim_{t\to 0} \inf\left\{ \sum_{j=1}^{\infty} h(\delta(B_j)) \mid D \subseteq \bigcup_{j=1}^{\infty} B_j, \delta(B_j) \leqslant t \right\},$$

where D is a Borel set, $\delta(B) = \sup\{d(x, y) \mid x, y \in B\}$ denotes the diameter of the set B, and h: $[0, \infty[\to [0, \infty[$ is a Hausdorff function (nondecreasing, $\lim_{t\to 0} h(t) = h(0) = 0$).

μ^h is a Borel measure ([**5**], cf. [**11, 2**]). Since we can require $B_j \subseteq D$ and $\delta(f(B_j)) = \delta(B_j)$ then holds for any isometry f: $D \to F$, μ^h is d-invariant.

The problem consists in choosing the function h in such a manner that $0 < \mu(A) < \infty$. In our case $X = G$, Kahnert [**7**] and Mycielski [**8, 9**] have suggested to take the functions $h_1(t) = 1/E(A, t)$ with $E(A, t) = \min\{k \mid A \subseteq \bigcup_{j=1}^{k} B_j, \delta(B_j) \leqslant t\}$ and $h_2(t) = \sup\{\lambda(B) \mid \delta(B) \leqslant t\}$.

It is easy to show that $\mu^{h_1}(A) \leqslant 1$ and $\mu^{h_2}(A) \geqslant \lambda(A)$ [**7**], but $\mu^{h_1}(A) = 0$ and $\mu^{h_2}(A) = \infty$ are possible [**1**, Examples 1 and 2], probably even at the same time, so that μ^h will hardly prove Theorem 1.

3. Fractional coverings. As a counterpart we examine the existence proofs for Haar measure. Haar's original method, as presented by Halmos [**4**, §58], is quite suitable for our purpose. If we replace $\lambda_U(C) = C : U/A : U$ by $\lambda_t(C) = E(C, t)/E(A, t)$ with $t > 0$ and $t \to 0$, Halmos's procedure will show immediately that λ is invariant with respect to isometries of G onto itself, and that open isometric subsets of G have equal measure (note that λ_t is d-invariant, and use [**4**, Theorems 53.F and 53.C]). However, the nonconstructive character of the proof makes it difficult to improve on these assertions.

Cartan, following Weil, constructed Haar measure as a linear form on a function space, using coverings $f \leqslant \sum c_j \cdot g_j$ of a positive function f by given functions g_j with coefficients $c_j \geqslant 0$ instead of coverings $D \subseteq \bigcup B_j$ of a set D (cf. [**6**, §15] or [**10**, Chapter II, §8]). This proof can hardly be connected with metrics, but the new coverings do apply successfully to Hausdorff measure.

Let 1_D, 1_{B_j} denote the characteristic functions of the sets D, B_j ($1_D(x) = 1$ for x in D, 0 otherwise). An expression of the form $1_D \leqslant \sum c_j \cdot 1_{B_j}$ with $c_j \geqslant 0$ is called a *fractional covering* of D by the sets B_j. Accordingly, the measure ν^h on a metric space (X, d) defined by

$$\nu^h(D) = \lim_{t\to 0} \inf\left\{ \sum_{j=1}^{\infty} c_j \cdot h(\delta(B_j)) \mid 1_D \leqslant \sum_{j=1}^{\infty} c_j \cdot 1_{B_j}, c_j \geqslant 0, \delta(B_j) \leqslant t \right\}$$

will be referred to as *fractional Hausdorff measure*.

As far as we know, this measure has been used first by Wegmann [**14**, Satz 3 and 4] as a tool for proving a theorem on Hausdorff dimension of products. In contemporary combinatorics, however, fractional coverings have already become popular: the relevant survey of Schrijver [**13**] comprises 175 references.

It is obvious that $\nu^h \leqslant \mu^h$. The fact that ν^h is a Borel measure can be verified as usual [**14, 5, 11, 2**]: ν^h is σ-subadditive and additive for sets with positive distance. As in §2, metric invariance of ν^h is easy to see.

4. Two Hausdorff functions. We return to our group $X = G$ and look for functions h with $0 < \nu^h(A) < \infty$. We shall see that $h_2(t) = \sup\{\lambda(B) \mid \delta(B) \leq t\}$ can be used as above, but h_1 has to be adapted to fractional coverings: Let $h_3(t) = 1/E'(A,t)$ with

$$E'(A,t) = \inf\left\{ \sum_{j=1}^{n} c_j \mid n \in N,\ 1_A \leq \sum_{j=1}^{n} c_j \cdot 1_{B_j},\ c_j \geq 0,\ \delta(B_j) \leq t \right\}.$$

PROPOSITION. (a) $\nu^{h_2}(D) \geq \lambda(D)$ *for all Borel sets D in G.*
(b) *If* $\underline{\lim}_{t\to 0} h_2(t)/h_3(t) < \infty$, *then* $0 < \nu^{h_2}(A) < \infty$.

To prove (a) we apply the Haar integral [6, 10]. $1_D \leq \sum_{j=1}^{\infty} c_j \cdot 1_{B_j}$ implies

$$\lambda(D) = \int 1_D \, d\lambda \leq \int \sum c_j \cdot 1_{B_j} \, d\lambda = \sum c_j \cdot \lambda(B_j) \leq \sum c_j \cdot h_2(\delta(B_j)).$$

Thus all sums in the definition of $\nu^{h_2}(D)$ are $\geq \lambda(D)$, and so $\nu^{h_2}(D) \geq \lambda(D)$. In particular, $\nu^{h_2}(A) \geq \lambda(A) > 0$. For (b) it remains to show $\nu^{h_2}(A) < \infty$. We have a constant c and a sequence $(t_k)_{k\in N}$, $t_k > 0$, $t_k \to 0$, with $h_2(t_k)/h_3(t_k) < c$ for all k. In other words, $E'(A,t_k) < c/h_2(t_k)$. Thus there is a fractional covering $1_A \leq \sum_{j=1}^{n_k} c_j^k \cdot 1_{B_j^k}$ with $c_j^k \geq 0$, $\delta(B_j^k) \leq t_k$ and $\sum c_j^k \leq c/h_2(t_k)$. Using this covering in the definition of $\nu^{h_2}(A)$ for $t = t_k$ we get $\inf\{\sum \cdots \leq t_k\} \leq \sum c_j^k \cdot h_2(t_k) \leq c$. Since this holds for all k, the limit $\nu^{h_2}(A)$ is also $\leq c$.

5. Cartan's approximation theorem for sets. The following statement says that "there are very economical fractional coverings by translations of any given open set". For simple coverings no similar assertion is possible, not even in Euclidean R^n [12, Chapter 8].

THEOREM 2. *Let (G, d) be a locally compact metric group, where d is left invariant, A a compact set in G, λ a left Haar measure and $\varepsilon > 0$. Then there is a number $t_0 > 0$ with the following property. For any open set B in G with $\delta(B) \leq t_0$ there exist points $s_1, \ldots, s_n$ in G and positive numbers $d_1, \ldots, d_n$ such that $\sum d_i \cdot 1_{s_i \cdot B} \geq 1_A$ and $\sum d_i \cdot \lambda(s_i \cdot B) \leq (1+\varepsilon) \cdot \lambda(A)$.*[1]

(In all sums of this section, i runs from 1 to n.) With this theorem we shall complete the proof of Theorem 1. For a given $\varepsilon > 0$, choose $t \leq \min\{t_0(\varepsilon), \varepsilon\}$ such that h_2 is continuous in t. We find an open set B with $\delta(B) \leq t$ and $h_2(t) \leq (1+\varepsilon)^2 \cdot \lambda(B)$. (Let $t' \leq t$ with $h_2(t) \leq (1+\varepsilon)h_2(t')$, let C be a set with $\delta(C) \leq t'$ and $h_2(t') \leq (1+\varepsilon) \cdot \lambda(C)$, and let $B = \{x \mid d(x,C) < t - t'\}$.) By Theorem 2 we have a covering $1_A \leq \sum d_i \cdot 1_{s_i B}$ with $\lambda(B) \cdot \sum d_i \leq (1+\varepsilon) \cdot \lambda(A)$. Consequently,

$$\begin{aligned} h_2(t)/h_3(t) &= h_2(t) \cdot E'(A,t) \leq h_2(t) \cdot \sum d_i \\ &\leq (1+\varepsilon)^2 \lambda(B) \cdot \sum d_i \leq (1+\varepsilon)^3 \lambda(A). \end{aligned}$$

Since ε is arbitrary, this implies $\underline{\lim}_{t\to 0} h_2(t)/h_3(t) \leq \lambda(A)$. Our proposition applies, $\mu = \nu^{h_2}$ is a d-invariant measure with $0 < \mu(A) < \infty$, and Theorem 1 is proved.

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[1]In the case $\lambda(A) = 0$, $\sum d_i \lambda(s_i B) \leq \varepsilon$.

It remains to prove Theorem 2 which is in fact only a modification of the following result of Cartan (cf. [**6**, p. 189] or [**10**, p. 115]), the key to a constructive proof of existence of Haar measure.

CARTAN'S THEOREM. *Let G be a locally compact group, f a positive continuous function on G with compact support and $\alpha > 0$. Then there is a neighbourhood U_0 of the unit element e in G with the following property.*

For any positive continuous function g with support in U_0 there exist points $s_1, \ldots, s_n$ in the support of f and positive numbers $c_1, \ldots, c_n$ such that

$$\left|f(x) - \sum c_i \cdot g\left(s_i^{-1} \cdot x\right)\right| \leqslant \alpha \quad \textit{for all } x \textit{ in } G.$$

To derive Theorem 2 from this result, let $c = \sqrt[3]{1+\varepsilon}$. Since λ is regular and A is compact, we find a neighbourhood $V = \{x \mid d(x, A) < b\}$ of A such that $\bar{V}$ is compact and $\lambda(V) \leqslant c \cdot \lambda(A)$. Let $W = \{x \mid d(x, A) < b/2\}$, and let $f: G \to [0, 1]$ be a continuous function with $f(A) = \{1\}$ and $f(G - W) = \{0\}$. In other words, $1_A \leqslant f \leqslant 1_W$. Choose $\alpha > 0$ such that $\alpha(1 + c) < c - 1$ and hence $(1 + \alpha)/(1 - \alpha) < c$. Applying Cartan's theorem to f and α we obtain U_0. Let us define $t_0 = \min\{b/2, d(e, G - U_0)\}$, and let us prove that t_0 has the desired property.

Let B be open and $\delta(B) \leqslant t_0$. Since $x^{-1} \cdot B \subseteq U_0$ for every x in B, we shall just assume that $e \in B \subseteq U_0$. By regularity of λ, there is a closed set $C \subseteq B$ with $c \cdot \lambda(C) \geqslant \lambda(B)$. Let $g: G \to [0, 1]$ be a continuous function with $1_C \leqslant g \leqslant 1_B$.

Since the support of g is in U_0, Cartan's theorem says that there are points s_i in W and numbers $c_i \geqslant 0$ with

$$(*) \qquad 1_A(x) - \alpha \leqslant f(x) - \alpha \leqslant \sum c_i \cdot g\left(s_i^{-1} \cdot x\right) \leqslant f(x) + \alpha \quad \text{for } x \text{ in } G.$$

We write $g_i(x) = g(s_i^{-1} \cdot x)$ and $d_i = c_i/(1 - \alpha)$. The definition of g and g_i implies $g_i \leqslant 1_{s_i B}$ and our first assertion

$$1_A \leqslant \sum d_i \cdot g_i \leqslant \sum d_i \cdot 1_{s_i B}.$$

The function on the right-hand side is zero on $G - V$ since the s_i are in W and $d(W, G - V) > b/2 \geqslant \delta(B)$. Thus $(*)$ implies $\sum d_i \cdot g_i \leqslant c \cdot 1_V$. By the properties of Haar integral,

$$\sum d_i \cdot \int g \, d\lambda = \sum d_i \cdot \int g_i \, d\lambda = \int \sum d_i \cdot g_i \, d\lambda \leqslant \int c \cdot 1_V \, d\lambda = c \cdot \lambda(V).$$

Using the definitions of C, g, V and c we finish the proof.

$$\sum d_i \cdot \lambda(s_i B) = \lambda(B) \cdot \sum d_i \leqslant c \cdot \lambda(C) \cdot \sum d_i \leqslant c \cdot \int g \, d\lambda \cdot \sum d_i$$
$$\leqslant c^2 \cdot \lambda(V) \leqslant c^3 \cdot \lambda(A) = (1 + \varepsilon) \cdot \lambda(A).$$

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