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STOKES' THEOREM AND PARABOLICITY OF RIEMANNIAN MANIFOLDS

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ABSTRACT. A noncompact Riemannian n-manifold is parabolic if and only if Stokes' theorem is valid for every square integrable $(n-1)$-form with integrable derivative.

For a compact orientable n-manifold R Stokes' theorem implies that

$$\int_R d\alpha = 0 \tag{1}$$

for every differentiable $(n-1)$-form α on R. In case R is an open relatively compact subset of a Riemannian n-manifold Bochner [**1**] established (1) for $(n-1)$-forms α vanishing "in average" at the boundary of R with $d\alpha$ integrable. Gaffney [**4**] extended (1) in a different direction by showing that it is valid when R is a complete Riemannian manifold and both α, $d\alpha$ are integrable. Subsequently Yau [**9**] established a weak form of (1) without any integrability assumptions on $d\alpha$. Recently Karp [**7**] showed that (1) holds for complete Riemannian manifolds satisfying certain volume growth conditions and α satisfying certain integrability conditions but $d\alpha$ merely nonnegative outside a compact set. Results of this sort have been labeled Stokes' theorem for noncompact manifolds.

The requirement that R be complete excludes from consideration many parabolic Riemannian manifolds (cf. [**8**]). A compact Riemannian manifold with countably many points deleted is an example of an incomplete parabolic manifold and is included in Bochner's result. Since parabolic Riemannian manifolds resemble compact ones from many points of view, it is natural to try to find conditions on α which imply (1) for parabolic R. The purpose of this note is to show that if R is a noncompact Riemannian n-manifold, then (1) holds for every square integrable $(n-1)$-form α with $d\alpha$ integrable precisely when R is parabolic.

We begin by fixing terminologies. Let R be a noncompact Riemannian n-manifold and $\{R_k\}_0^\infty$ an exhaustion of R by relatively compact regions with smooth boundaries. Consider $\{w_k\}_1^\infty$ a sequence of continuous piecewise differentiable functions on R with $w_k \mid \bar{R}_0 = 1$, $w_k \mid R_k \setminus \bar{R}_0$ harmonic, $w_k \mid R \setminus R_k = 0$. Obviously, $w_k \leq w_{k+1} \leq 1$ and therefore $w = \lim w_k$ exists on R. Moreover, w is harmonic on $R \setminus \partial R_0$ and superharmonic on R.

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For suitable functions φ, ψ on R the mixed Dirichlet integral is given by $D_R(\varphi, \psi) = \int_R d\varphi \wedge * d\psi$ and the Dirichlet integral of φ by $D_R(\varphi) = D_R(\varphi, \varphi)$. We claim that $\{w_k\}$ converges to w in the Dirichlet seminorm as well. To this end note that by Green's formula $D_R(w_{p+k} - w_k, w_{p+k}) = 0$ for every pair of positive integers p, k. Thus $0 \leq D_R(w_{p+k} - w_k) = D_R(w_k) - D_R(w_{p+k})$. This implies that $d = \lim_k D_R(w_k)$ exists. Thus letting $p \uparrow +\infty$ and applying Fatou's lemma gives $D_R(w - w_k) \leq D_R(w_k) - d$, which establishes the claim.

The manifold R is called *parabolic* if w is identically 1. In view of the above, the parabolicity of R is equivalent to $d = 0$. Several other characterizations of parabolicity are used, for example, the nonexistence of a global Green's function, the nonexistence of nonconstant negative subharmonic functions or the validity of the boundary maximum principle (cf. **[5]**).

Let $\Gamma^{n-1}(R)$ denote the space of square integrable $(n-1)$-forms on R, i.e. $\int_R \alpha \wedge * \alpha < +\infty$ for $\alpha \in \Gamma^{n-1}(R)$. Also set $\Theta^{n-1}(R) = \{\alpha \in \Gamma^{n-1}(R);\ \int_R |d\alpha| < +\infty\}$, where $|d\alpha| = |* d\alpha| * 1$.

THEOREM. *The Riemannian n-manifold R is parabolic if and only if* (1) *holds for every* $\alpha \in \Theta^{n-1}(R)$.

Assume that R is parabolic and let $\alpha \in \Theta^{n-1}(R)$. Then the sequence $\{w_k\}$ has the properties

$$w_k \uparrow 1 \quad \text{on } R \tag{2}$$

and

$$D_R(w_k) \downarrow 0. \tag{3}$$

By (2) and the Lebesgue dominated convergence theorem we have

$$\lim_k \int_{R \setminus R_0} w_k\, d\alpha = \int_{R \setminus R_0} d\alpha. \tag{4}$$

For an arbitrary positive integer k we have

$$\int_{R \setminus R_0} w_k\, d\alpha = \int_{R \setminus R_0} d(w_k \alpha) + \int_{R \setminus R_0} dw_k \wedge \alpha. \tag{5}$$

Using the usual Stokes' theorem we see that $\int_{R \setminus R_0} d(w_k\alpha) = -\int_{\partial R_0} \alpha = -\int_{R_0} d\alpha$ and thus by (5)

$$\int_{R_0} d\alpha + \int_{R \setminus R_0} w_k\, d\alpha = \int_{R \setminus R_0} dw_k \wedge \alpha. \tag{6}$$

By the Schwarz inequality the absolute value of the right side of (6) is bounded by $(D_R(w_k) \int_R \alpha \wedge * \alpha)^{1/2}$. Consequently, (3) implies that the limit of the left side is 0 as $k \uparrow +\infty$. This together with (4) establish the necessity of (1).

Conversely, assume that (1) holds for every $\alpha \in \Theta^{n-1}(R)$. Fix a compact neighborhood N of ∂R_0. Since w is superharmonic on R and $w \mid R \setminus \partial R_0$ is harmonic, we may choose a sequence $\{s_j\}$ of C^2 superharmonic functions on R such that $\lim s_j = w$ on R and s_j agrees with w on $R \setminus N$. (The usual proof of this approximation by C^2 superharmonic functions given in **[6]** can be adapted to Riemannian manifolds by using the mean value property established by Feller [3].) Since $D_R(s_j) < +\infty$ and $d{*}\,ds_j = 0$ on $R \setminus N$, we have $*\,ds_j \in \Theta^{n-1}(R)$. Thus $\int_R d{*}\,ds_j = 0$. In view of $*\,d*ds_j \leqslant 0$ on R we see that s_j is harmonic on R and consequently w is harmonic on R. Since w achieves its maximum, it is constant. This means that R is parabolic.

From this theorem, or from the other characterizations of parabolicity mentioned above, it is obvious that the notion of parabolicity is independent of the choice of the exhaustion $\{R_k\}$. A slight modification of the proof gives the following

COROLLARY. *A noncompact Riemannian n-manifold R is parabolic if and only if* (1) *holds for every* $\alpha \in \Gamma^{n-1}(R)$ *such that there exists a compact set* K_α *with* $*\,d\alpha \geqslant 0$ *on* $R \setminus K_\alpha$.

Indeed, if R is parabolic and we are given such an α, then we may choose the exhaustion $\{R_k\}_0^\infty$ with $K_\alpha \subset R_0$. Then (2) is valid and since $*\,d\alpha \geqslant 0$ on $R \setminus R_0$, the monotone convergence theorem implies that (4) holds. The remainder of the proof is the same as that of the theorem.

On a parabolic Riemannian manifold every Dirichlet finite harmonic function is constant (cf. [5]). Using this we obtain the

COROLLARY. *A noncompact Riemannian manifold R is parabolic if and only if every* C^2 *Dirichlet finite subharmonic function on R is constant.*

According to a result of Cheng and Yau [2] a complete Riemannian manifold with second order volume growth is parabolic. Combining this with our first corollary gives a special case of Karp's [7] Corollary 1.

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