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CLOSED OPERATORS AND PURE CONTRACTIONS IN HILBERT SPACE[1]

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ABSTRACT. Some properties of the one-to-one mapping $A \to A(1 - A^*A)^{-1/2}$ of the pure contractions onto the closed and densely-defined operators are proved, in particular that it commutes with adjunction and preserves normality.

In a Hilbert space H, a *pure contraction* is a linear operator A on H such that $\|Ax\| < \|x\|$ for all $x \neq 0$ in H. In [2] I showed that the function Γ defined by $\Gamma(A) = A(1 - A^*A)^{-1/2}$ maps the set of all pure contractions one-to-one onto the set of all closed and densely-defined operators in H. Since Γ preserves many properties of operators, it may be used to reformulate questions about unbounded operators in terms of bounded ones. For example, in [1, §5, pp. 708–712], Cordes and Labrousse prove that if a closed and densely-defined operator C is semi-Fredholm then so is the bounded operator $C(1 + C^*C)^{-1/2}$; the latter is simply $\Gamma^{-1}(C)$.

The purpose of the present report is to indicate some further properties of Γ, and to establish some connections between unbounded symmetric operators and certain hyponormal contractions, leading to a reduction of the problem of finding selfadjoint extensions of symmetric operators to a corresponding problem involving only bounded operators.

From this point on, A denotes a pure contraction, B and B_* the associated *defect operators* $(1 - A^*A)^{1/2}$ and $(1 - AA^*)^{1/2}$, respectively, and C the closed and densely-defined operator $\Gamma(A) = AB^{-1}$. (Note that $\Gamma(A^*) = A^*B_*^{-1}$.) We take for granted the following relations proved in [2]: $\operatorname{ran} B = \operatorname{dom} C$, $C^* = B^{-1}A^*$, $B = (1 + C^*C)^{-1/2}$, and (thus) $C^*C = B^{-2} - 1$. Note also that since A is a pure contraction, B and B^* are one-to-one.

THEOREM 1. *The operator C has an everywhere-defined and bounded inverse if and only if the operator A is invertible.*

PROOF. Since B is one-to-one and $\operatorname{ran} A = \operatorname{ran} C$, A is invertible if and only if C is one-to-one with range H. By the closed graph theorem, this is equivalent to C having a bounded inverse with domain H.

LEMMA 1. *Γ commutes with adjunction, i.e.,* $\Gamma(A^*) = \Gamma(A)^*$.

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PROOF. It is well known that $A^*B_* = BA^*$ and, dually, $AB = B_*A$ (e.g. see [**3**, pp. 6–7]). Hence

$$\Gamma(A)^*|_{\text{ran } B_*} = B^{-1}A^*B_*B_*^{-1} = B^{-1}BA^*B_*^{-1} = \Gamma(A^*).$$

To complete the proof, we show that dom $\Gamma(A)^* \subseteq$ ran B_*, i.e., if A^*x is in ran B then x is in ran B_*. Let $A^*x = By$; then $(1 - B_*^2)x = AA^*x = ABy = B_*Ay$ and thus $x = B_*(B_*x - Ay)$, as was to be proved.

LEMMA 2. *If $AB = BA$ then $AB^{-1} = B^{-1}A$.*

PROOF. We modify the proof of Lemma 1: if $AB = BA$ then $B^{-1}ABB^{-1} = B^{-1}BAB^{-1}$, so that $B^{-1}A|_{\text{ran } B} = AB^{-1}$. Let $Ax = By$; then

$$(1 - B^2)x = A^*Ax = A^*By = BA^*y$$

(this last since $A^*B = BA^*$). Hence $x = B(Bx - A^*y)$. Therefore $B^{-1}A = AB^{-1}$.

DEFINITION. The operator C is *quasinormal* provided that $C(C^*C) = (C^*C)C$.

THEOREM 2. *Both Γ and Γ^{-1} preserve normality and quasinormality. I.e., C is normal (resp. quasinormal) if and only if A is normal (resp. quasinormal).*

PROOF. We first observe that by Lemma 1 and the fact that Γ is one-to-one, $\Gamma^{-1}(C^*) = A^*$ and thus $B_* = (1 + CC^*)^{-1/2}$. From this, the equivalence of the normality of A and C may be seen as follows: A is normal if and only if $(1 - A^*A)^{1/2} = (1 - AA^*)^{1/2}$ (i.e., $B = B_*$), which holds only in case $(1 + C^*C)^{1/2} = (1 + CC^*)^{1/2}$, which in turn is true if and only if $C^*C = CC^*$.

Now suppose A is quasinormal. Then $AB = BA$ and therefore by Lemma 2, $C = AB^{-1} = B^{-1}A$. Hence (recall that $C^*C = B^{-2} - 1$) we get that

$$C(C^*C) = AB^{-1}(B^{-2} - 1) = B^{-1}AB^{-2} - AB^{-1} = B^{-2}AB^{-1} - AB^{-1} = (C^*C)C.$$

Conversely, C quasinormal implies $CB = BC$ and therefore $AB = CB^2 = B(CB) = BA$, from which it follows at once that A is quasinormal.

LEMMA 3. $\Gamma(|A|) = |C|$, *and if $V|A|$ is the polar decomposition of A then $V|C|$ is the polar decomposition of C.*

PROOF. By definition, $|T| = (T^*T)^{1/2}$. We calculate

$$|C| = (B^{-2} - 1)^{1/2} = (1 - B^2)^{1/2}B^{-1} = (A^*A)^{1/2}B^{-1} = \Gamma(|A|),$$

and thus $V|C| = V|A|B^{-1} = AB^{-1} = C$.

REMARKS. By Lemma 1, Γ preserves selfadjointness (if $A = A^*$ then $C = \Gamma(A) = \Gamma(A^*) = \Gamma(A)^* = C^*$); Lemma 3 may be viewed as stating that Γ "preserves argument." With the usual identification of the complex numbers as scalar multiplications on H, Γ^{-1} is seen to be an extension of the radial compression $z \to z(1 + |z|^2)^{-1/2}$ of the complex plane onto the unit circular disc. As we proved in [**2**], the "extended compression" Γ^{-1} throws the bounded operators on H onto the *interior* of the unit ball in the algebra of all bounded operators on H, whereas each unbounded closed and densely-defined operator is mapped to its *boundary*. Except for the factor $\sqrt{2}$, the unit circle in the plane is left invariant under Γ^{-1}. The following result characterizes the invariants of $\sqrt{2}\,\Gamma^{-1}$:

THEOREM 3. *An operator is invariant under $\sqrt{2}\,\Gamma^{-1}$ if and only if it is a partial isometry.*

PROOF. Suppose C is a partial isometry. Then $|C|$ is a projection, say P. Hence by Lemma 3, with V as indicated therein,

$$A = V\Gamma^{-1}(P) = VP(1+P)^{-1/2} = VP(1-P+2P)^{-1/2}$$
$$= VP(1-P) + V\sqrt{\tfrac{1}{2}}\,P = \sqrt{\tfrac{1}{2}}\,C.$$

Conversely, suppose $\sqrt{2}A = C$. Then $B^{-1}C^* = B^{-1}\sqrt{2}A^* = \sqrt{2}\,C^*$; i.e., $B^{-1} = \sqrt{2}$ on $\operatorname{ran} C^*$. Hence for all x in $\operatorname{ran} C^*$,

$$\|Cx\|^2 - \|x\|^2 = \|x\|^2 + \|Cx\|^2 - 2\|x\|^2 = \|(1+C^*C)^{1/2}x\|^2 - \|\sqrt{2}\,x\|^2$$
$$= \|B^{-1}x\|^2 - \|\sqrt{2}\,x\|^2 = 0.$$

This confirms that the restriction of C to $\operatorname{ran} C^*$ is an isometry, which implies that C is a partial isometry.

THEOREM 4. *C is symmetric if and only if A^*B is selfadjoint, in which case A is hyponormal.*

PROOF. Let $\langle\cdot,\cdot\rangle$ denote the inner product in H. Then C is symmetric if and only if, for all (x, y) in $H \times H$, $\langle CBx, By\rangle = \langle Bx, CBy\rangle$; since $A = CB$, this is equivalent to $\langle x, A^*By\rangle = \langle A^*Bx, y\rangle$, which is simply the selfadjoint condition on A^*B. Thus C is symmetric only in case A^*B is selfadjoint. Now suppose that this is indeed the case: for all x in $\operatorname{ran} B$ ($= \operatorname{dom} C$),

$$\|B_*^{-1}x\|^2 = \|(1+CC^*)^{1/2}x\|^2 = \|x\|^2 + \|C^*x\|^2$$
$$= \|x\|^2 + \|Cx\|^2 = \|B^{-1}x\|^2.$$

This establishes that $B_*^{-1}B$ is an isometry on H and therefore $B_*^{-1}B^2B_*^{-1} \subseteq (B_*^{-1}B)(B_*^{-1}B)^* \leqslant 1$, so that $B^2 \leqslant B_*^2$. Hence $AA^* = 1 - B_*^2 \leqslant 1 - B^2 = A^*A$, which is to say that A is hyponormal.

As a further example of applying facts about Γ to questions concerning unbounded operators, there is the following:

THEOREM 5. *If a closed and densely-defined quasinormal operator is symmetric then it is selfadjoint.*

PROOF. If C is quasinormal then by Theorem 2, $A = \Gamma^{-1}(C)$ is also. If C is symmetric then by Theorem 4, $A^*B = (A^*B)^* = BA$. But since A is quasinormal, $BA = AB$; thus A^* agrees with A on the dense subspace $\operatorname{ran} B$ of H. Hence $A = A^*$ and thus by Lemma 1, $C = C^*$.

For the remainder of this report, we focus on the relationship between C and AB_* when C is symmetric. Note that in this case if AB_* is selfadjoint then by Theorem 4, C^* is symmetric, so that C is selfadjoint. We will show that the search for selfadjoint extensions of C is reducible to the finding of *maximal* linear subspaces M of H on which the bounded operator AB_* is symmetric, in the sense that for all x and y in M, $\langle AB_*x, y\rangle = \langle x, AB_*y\rangle$. We start with a technical lemma.

LEMMA 4. *If C is symmetric then $A^2 + B_*B$ is the isometry $B_*^{-1}B$ from H onto $B_*^{-1}(\operatorname{dom} C)$, and $(AB_* - B_*A)(A^2 + B_*B) = 0$.*

PROOF. We recall from the proof of Theorem 4 the fact that $B_*^{-1}B$ is an isometry; clearly its range is $B_*^{-1}(\operatorname{dom} C)$. Using the relations $C^*B = CB = A$ (C is symmetric) and $AA^* + B_*^2 = 1$, we get that

$$B_*^{-1}B = AA^*B_*^{-1}B + B_*^2B_*^{-1}B = AC^*B + B_*B = A^2 + B_*B.$$

The rest of the lemma is now available by direct calculation:

$$\begin{aligned}\left(AB_* - B_*A^*\right)\left(A^2 + B_*B\right) &= \left(AB_* - B_*A^*\right)B_*^{-1}B = AB - B_*C^*B \\ &= AB - B_*CB = AB - B_*A = 0.\end{aligned}$$

THEOREM 6. *If C is symmetric then an operator T is a symmetric extension of C if and only if $T = C^*|_{B_*(M)}$, where M is a linear subspace of H containing $\operatorname{ran}(A^2 + B_*B)$ and on which AB_* is symmetric. In this case, T is closed if and only if M is closed.*

PROOF. Note that the equation $(AB_* - B_*A^*)(A^2 + B_*B) = 0$ of Lemma 4 assures us that AB_* is automatically symmetric on $\operatorname{ran}(A^2 + B_*B)$. It is elementary that an operator T is a symmetric extension of C only in case $T = C^*|_S$, where S is a linear subspace of $\operatorname{dom} C^*$ including $\operatorname{dom} C$, and on which C^* is symmetric. Now $\operatorname{dom} C^* = \operatorname{ran} B^*$ and, since $C \subseteq C^*$ (so that $\operatorname{ran} B \subseteq \operatorname{ran} B^*$), we know also that $\operatorname{dom} C = \operatorname{ran} B_*B_*^{-1}B = B_*(\operatorname{ran} B_*^{-1}B)$. By Lemma 4, $B_*^{-1}B = A^2 + B_*B$; hence $\operatorname{dom} C = B_*(\operatorname{ran}(A^2 + B_*B))$. From this it is clear that the subspace M promised by the theorem is simply $B_*^{-1}(S)$, and M includes $\operatorname{ran}(A^2 + B_*B)$ because S includes $\operatorname{dom} C$. All that remains is to verify that the symmetry of C^* on $S = B_*(M)$ is equivalent to the symmetry of AB_* on M, viz.: for all (x, y) in $M \times M$, $\langle AB_*x, y\rangle = \langle x, AB_*y\rangle$ if and only if $\langle B_*x, C^*(B_*y)\rangle = \langle C^*(B_*x), B_*y\rangle$, since $A^* = C^*B_*$.

Suppose now that T is the symmetric extension $C^*|_{B_*(M)}$ of C and consider: M is closed if and only if $B_*(M)$ is closed with respect to the norm $\|B_*^{-1}\cdot\|$ on $\operatorname{ran} B_*$. Since for all x in $\operatorname{dom} C^* = \operatorname{ran} B_*$, the norm in $H \times H$ of the member (x, C^*x) of C^* is given by $\|B_*^{-1}x\| = \|(1 + CC^*)^{1/2}x\| = (\|x\|^2 + \|C^*x\|^2)^{1/2}$, we conclude that M is closed if and only if T is closed, and the proof is complete.

REMARK. Since the equations $T = C^*|_{B_*(M)}$ define an inclusion-preserving mapping from the symmetric extensions T of C onto certain subspaces M of H, one might inquire as to whether there is available a natural adjunction for such subspaces M. This is indeed the case.

THEOREM 7. *If C is symmetric and $T = C^*|_{B_*(M)}$ is a symmetric extension of C, then $T^* = C^*|_{B_*(M^*)}$, where $M^* = (AB_* - B_*A^*)(M)^\perp$.*

PROOF. For all x in H, B_*x is in $\operatorname{dom} T^*$ if and only if for all y in M, $\langle B_*x, C^*B_*y\rangle = \langle C^*B_*x, B_*y\rangle$; equivalently,

$$\begin{aligned}\langle x, \left(AB_* - B_*A^*\right)y\rangle &= \langle A^*x, B_*y\rangle - \langle B_*x, A^*y\rangle \\ &= \langle C^*B_*x, B_*y\rangle - \langle B_*x, C^*B_*y\rangle = 0\end{aligned}$$

But the requirement that $\langle x, (AB_* - B_*A^*)y\rangle = 0$ for all y in M is just the requirement that x be in $(AB_* - B_*A^*)(M)^\perp = M^*$. Hence dom $T^* = B_*(M^*)$, as was to be proved.

COROLLARY. *The extension T is selfadjoint if and only if $M = M^*$, i.e., M is a maximal subspace of H on which AB_* is symmetric.*

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