PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 87, Number 1, January 1983

ON A GENERALIZED MOMENT PROBLEM

J. S. HWANG

ABSTRACT. The well-known Müntz-Szász theorem asserts that the sequence of powers x^{n_p} is complete on $[a, b]$, where $a \geq 0$, if and only if

$$(1) \qquad \sum_{p=1}^{\infty} \frac{1}{n_p} = \infty, \quad \text{where } 0 < n_1 < n_2 < \cdots.$$

Let $f(x)$ be absolutely continuous, $|f'(x)| \geq k > 0$, and $f(a)f(b) \geq 0$. We prove that under the assumption (1) the sequence $\{f(x)^{n_p}\}$ is complete on $[a, b]$ if and only if $f(x)$ is monotone on $[a, b]$.

1. Introduction. Let $L[a, b]$ be the space of all summable functions defined on the finite interval $[a, b]$. As usual (see R. P. Boas [**2**, p. 234]), a sequence of functions $f_n(x)$ is complete on $[a, b]$ if for any $g \in L[a, b]$, the equalities

$$\int_a^b f_n(x)g(x)\,dx = 0, \qquad n = 1, 2, \ldots,$$

imply that $g(x) = 0$, a.e. (almost everywhere) on $[a, b]$.

The Müntz-Szász theorem can be extended as follows by elementary methods.

THEOREM 1. *Under the assumption* (1), *the sequence of polynomials* $(x(1-x))^{n_p}$ *is complete on* $[0, \delta]$ *if and only if* $\delta \leq \frac{1}{2}$.

Naturally, we may ask whether the polynomial $x(1-x)$ can be replaced by a more general function. In this note, we settle such an extension of the Müntz-Szász theorem as well as a related theorem of Picone; see Boas [1].

2. Müntz-Szász theorem. To formulate the desired theorem, we first observe that the derivative of the function $f(x) = x(1-x)$ satisfies $f'(x) = 1 - 2x \geq 1 - 2\delta > 0$ for every $x \in [0, \delta]$, where $\delta < \frac{1}{2}$. This gives the motivation for the following extension of the Müntz-Szász theorem.

THEOREM 2. *Let* $f(x)$ *be a function absolutely continuous on a finite interval* $[a, b]$, $f(a)f(b) \geq 0$, *and* $|f'(x)| \geq k > 0$, *a.e. on* $[a, b]$; *then under the assumption* (1) *the sequence* $\{f(x)^{n_p}\}$ *is complete on* $[a, b]$ *if and only if* $f(x)$ *is monotone on* $[a, b]$.

PROOF. *Sufficiency.* We may, without loss of generality, assume that the function $y = f(x)$ is montonically increasing on $[a, b]$. Let $x = f^{-1}(y)$ be the inverse function

Received by the editors December 2, 1980 and, in revised form, March 31, 1981.

1980 *Mathematics Subject Classification.* Primary 30B60, Secondary 26A48.

Key words and phrases. Completeness, moment problem, absolutely continuous and monotone function.

[1] The author acknowledges support from H. L. Jackson's NSERC Grant (A7322) when the author visited McMaster University.

of $f(x)$; this function is montonically increasing on $[f(a), f(b)]$. Then by the identity $y = f(f^{-1}(y))$ and the hypothesis $|f'(x)| \geqslant k > 0$, a.e. on $[a, b]$, we obtain

$$|f^{-1}(y)'| = 1/|f'(x)| \leqslant 1/k, \quad \text{a.e. on } [f(a), f(b)]. \tag{2}$$

Moreover, $f(x)$ is absolutely continuous so that its derivative $f'(x)$ is summable on $[a, b]$. It follows from the left-hand side of (2) that

$$f^{-1}(y)' \neq 0, \quad \text{a.e. on } [f(a), f(b)]. \tag{3}$$

We now consider an arbitrary function $g \in L[a, b]$ for which

$$\int_a^b f(x)^{n_p} g(x)\,dx = 0, \qquad p = 1, 2, \ldots. \tag{4}$$

By substituting $y = f(x)$ into (4), we find that

$$\int_{f(a)}^{f(b)} y^{n_p} g(f^{-1}(y))(f^{-1}(y))'\,dy = 0, \qquad p = 1, 2, \ldots.$$

In view of (2), we can see that the function $g(f^{-1}(y))(f^{-1}(y))'$ is summable on $[f(a), f(b)]$. It follows from the Müntz-Szász theorem that this function

$$g(f^{-1}(y))(f^{-1}(y))' = 0, \qquad \text{a.e. on } [f(a), f(b)].$$

Owing to (3), we conclude that $g(x) = 0$, a.e. on $[a, b]$. This proves the completeness of the sequence $\{f(x)^{n_p}\}$ on $[a, b]$.

Necessity. If the sequence $\{f(x)^{n_p}\}$ is complete on $[a, b]$ we shall prove that $f(x)$ is monotone on $[a, b]$. Suppose not; then clearly there exist points $q < r$ in $[a, b]$ such that

$$f(q) = f(r). \tag{5}$$

We then define the function

$$\begin{cases} g(x) = f'(x), & \text{for } q < x < r \text{ and } f'(x) \text{ exists,} \\ = 0, & \text{elsewhere.} \end{cases} \tag{6}$$

By (5) and (6) we obtain for $p = 1, 2, \ldots$ (integration by parts)

$$\int_a^b f(x)^{n_p} g(x)\,dx = \int_q^r f(x)^{n_p} f'(x)\,dx = -\int_q^r n_p f(x)^{n_p} f'(x)\,dx,$$

or

$$(1 + n_p)\int_a^b f(x)^{n_p} g(x)\,dx = 0.$$

Since $1 + n_p \neq 0$, it follows from (6) that the sequence $\{f(x)^{n_p}\}$ is not complete on $[a, b]$, a contradiction. This completes the proof.

We finally remark that the uniqueness theorem of Boas [1] and Mikusiński [3] can also be extended as in Theorem 2.

References

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Institute of Mathematics, Academia Sinica, Taipei, Taiwan

Current address: Department of Mathematics, Michigan State University, East Lansing Michigan 48824