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ON THE OPERATOR RANGES OF ANALYTIC FUNCTIONS

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ABSTRACT. Following Doob, we say that a function $f(z)$ analytic in the unit disk U has the property $K(\rho)$ if $f(0) = 0$ and for some arc γ on the unit circle whose measure $|\gamma| \geqslant 2\rho > 0$,

$$\liminf_{j \to \infty} |f(z_j)| \geqslant 1 \quad \text{where } z_j \to z \in \gamma \text{ and } z_j \in U.$$

Let H be a Hilbert space over the complex field, A an operator whose spectrum is included in U, $\|A\|$ the operator norm of A, and $f(A)$ the usual Riesz-Dunford operator. We prove that there is no function with the property $K(\rho)$ satisfying

$$(1 - \|A\|)\|f'(A)\| \leqslant 1/n \quad \text{for all } \|A\| < 1,$$

where $n > N(\rho) = \log(1/(1 - \cos\rho))$. We also show that if f has the property $K(\rho)$ then the operator range of $f(A)$ covers a ball of radius $k(\rho) = \sqrt{3}/(4N(\rho))$. These two results generalize our previous solutions of two long open problems of Doob [**1**]. Finally, we prove that the operator range of any 4-fold univalent function is not convex. This extends our solution to Ky Fan's Problem [**4**].

1. Introduction. Let $f(z)$ be a function analytic in the unit disk $U = \{z: |z| < 1\}$. Following Doob [**1**, p. 119], we say that a function $f(z)$ has the property $K(\rho)$ if $f(0) = 0$ and for some arc γ on the unit circle of measure $|\gamma| \geqslant 2\rho > 0$,

$$\liminf_{j \to \infty} |f(z_j)| \geqslant 1,$$

where $\{z_j\}$ is an arbitrary sequence of points in U converging to an arbitrary interior point of γ.

In our recent works [**6–10**], we have solved two long open problems of Doob [**1**, p. 120]. In particular, in [**8–10**], we proved both of the following theorems:

THEOREM 1. *There is no function with the property $K(\rho)$ satisfying*

$$(1 - |z|)|f'(z)| \leqslant 1/n \quad \textit{for all } |z| < 1,$$

where $n > N(\rho) = \log(1/(1 - \cos\rho))$, $0 < \rho < \rho_0$ for some $\rho_0 < \pi/2$.

THEOREM 2. *If $f(z)$ has the property $K(\rho)$, then the range of $f(z)$ covers some simple and nonsimple disks of radii $k(\rho) = \sqrt{3}/(4N(\rho))$ and $1/(2N(\rho))$, respectively.*

In this paper, we shall extend these two theorems from the function theory to the operator theory. Let H be a Hilbert space over the complex field, A an operator (i.e. a bounded linear transformation) on H, and $\sigma(A)$ the spectrum of A. If $f(z)$ is a

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function analytic on a neighborhood G of $\sigma(A)$, then $f(A)$ will denote the Riesz-Dunford integral operator (see [**3**, p. 568])

$$f(A)=\frac{1}{2\pi i}\int_C f(z)(zI-A)^{-1}\,dz,$$

where I stands for the identity operator on H, and C is a positively oriented simple closed rectifiable contour such that the interior C° of C contains $\sigma(A)$ and the union $C\cup C^\circ\subset G$.

As usual, we denote the norm of an operator A by $\|A\|$ and we say that A is a contraction or a proper contraction if $\|A\|\leqslant 1$ or $\|A\|<1$, respectively. With this notion, we shall prove the following operator analogues of Theorems 1 and 2:

THEOREM 3. *There is no function with the property $K(\rho)$ satisfying*

$$(1-\|A\|)\|f'(A)\|\leqslant 1/n \quad \textit{for all } \|A\|<1,$$

where $n>N(\rho)=\log(1/(1-\cos\rho))$, $0<\rho<\rho_0$ for some $\rho_0<\pi/2$.

THEOREM 4. *If $f(z)$ has the property $K(\rho)$, then the operator range of $f(A)$ covers some ball of radius $k(\rho)=\sqrt{3}/(4N(\rho))$.*

Recently, Ky Fan [**4**, Theorem 7] proved that if $f(z)$ is a starlike function then its operator range $f(A)$ is also starlike. He then asked [**4**, p. 287] whether the operator range $f(A)$ is convex if $f(z)$ is a convex function, where $f(0)=0$ and $f'(0)=1$. In [**11**], we answered this question in the negative by the following Schwarz function (see [**5**, p. 385]) which is 4-fold ($\Sigma c_{4n+1}z^{4n+1}$):

$$s(z)=\int_0^z(1-t^4)^{-1/2}\,dt=z+\sum_{l}^{\infty}\frac{1\cdot 3\cdots(2n-1)}{2^n n!\,(4n+1)}z^{4n+1}.$$

We shall now extend this answer as follows:

THEOREM 5. *If $f(z)\neq z$ is a 4-fold univalent function ($f'(0)=1$), then the operator range of $f(A)$ is not convex.*

2. Proof of Theorem 3. According to Theorem 1, there is a point $z_0\in U$ such that

$$(1-|z_0|)|f'(z_0)|>1/n,\quad \text{where } n>N(\rho).$$

Let $A=z_0I$, then by the definition of the Riesz-Dunford operator, we get $f'(z_0I)=f'(z_0)I$, so that $\|f'(A)\|=|f'(z_0)|$ .It follows that

$$(1-\|A\|)\|f'(A)\|>1/n,\quad \text{where } n>N(\rho).$$

This proves Theorem 3.

3. Proof of Theorem 4. Let $f(z)$ have the property $K(\rho)$. Then by virtue of Theorem 2, we see that the function $f(z)$ is univalent between a subdomain $S\subset U$ and a disk of radius $k(\rho)$ with center, say, at $f(z_0)$, where $z_0\in S$. Let $g(w)$ be a conformal mapping from the unit disk U_w in the w-plane onto S such that $g(0)=z_0$. Then the function $h(w)=f(g(w))-f(z_0)$ is univalent on U_w, where $h(0)=f(g(0))-f(z_0)=0$.

Now, by letting the function $h_\rho(w) = k(\rho)w$, we have the following subordination:

$$k(\rho)U_w \subset h(U_w) \quad \text{or} \quad h_\rho(U_w) \subset h(U_w).$$

According to Fan's theorem [4, Theorem 7], for every proper contraction A on a Hilbert space H, there is a proper contraction B on H satisfying $h_\rho(A) = k(\rho)A = h(B)$. Since $\|A\| < 1$, this shows that the operator range of $h(B)$ covers the ball $\{A: \|A\| < \mathrm{k}(\rho)\}$.

In view of the spectral mapping theorem [3, p. 570], we find that $f(g(B)) = f(z_0)I + h(B)$, so that the operator range of the composite function $f \circ g$ covers the ball $\{A: \|f(z_0)I + A\| < k(\rho)\}$ with center at the point $-f(z_0)I$. Since the mapping $g(w)$ satisfies $|g(w)| < 1$ for all $w \in U_w$, it follows from Fan's theorem [4, Theorem 1] that $\|g(B)\| < 1$ for any $\|B\| < 1$. This in turn implies that the operator range of $f(A)$ covers that of $f \circ g$ over the same unit ball $\{A: \|A\| < 1\}$. We thus conclude that the operator range of $f(A)$ covers the desired ball $\{A: \|f(z_0)I + A\| < k(\rho)\}$. This completes the proof.

4. Proof of Theorem 5. The method here is an extension of [11]. Since the function f is 4-fold, it can be represented by

$$f(z) = z + \sum c_{4n+1} z^{4n+1} \neq z.$$

Let $A_1 = \zeta\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $A_2 = \zeta\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, where $|\zeta| = r < 1$.Then the norm $\|A_1\| = \|A_2\| = r < 1$. Since c_{2n} and c_{2n+1} are all zero, by a simple computation we find that

$$f(A_1) = f(\zeta)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad f(A_2) = f(\zeta)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

If the assertion were false, then there would be a proper contraction A such that

$$f(A) = \frac{1}{2}(f(A_1) + f(A_2)) = f(\zeta)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = B.$$

Since the function $w = f(z)$ is univalent and $f(0) = 0$, it follows that the inverse $z = f^{-1}(w)$ is analytic at the origin and can be expanded as $z = f^{-1}(w) = w + \Sigma a_n w^n$. This yields $A = f^{-1}(B) = B$, because $B^n = 0$ for $n > 1$, and the norm $\|A\| = \|B\| = |f(\zeta)|$, where $A(= B)$ depends on ζ.

We shall now show that there is a point $\zeta \in U$ for which the value $|f(\zeta)| > 1$. Suppose, on the contrary, that $|f(\zeta)| \leq 1$ for all $\zeta \in U$. Then by Schwarz's lemma (see [5, p. 236]), we have either $|f'(0)| < 1$ or $f(z) = e^{i\alpha}z$, contradicting the normalization $f'(0) = 1$ and the assumption $f(z) \neq z$, respectively. We thus prove that $|f(\zeta)| > 1$ for some $\zeta \in U$. This in turn implies that the norm $\|A\| = |f(\zeta)| > 1$, a contradiction. Hence, the operator range of $f(X)$ is not convex and the proof is complete.

5. Generalization. Instead of the Doob norm considered in Theorem 3, we can also consider the Bloch norm (see Hwang and Rung [6, 7]). In this connection, by the same argument, we can easily obtain the following two results analogous to Theorems 3 and 4:

THEOREM 6. *There is no function with the property $K(\rho)$ satisfying*

$$(1 - \|A^2\|)\|f'(A)\| \leq 1/n \quad \text{for all } \|A\| < 1,$$

where $n > N_1(\rho) = e(\pi - \rho)/(2\sin\rho)$, $0 < \rho < \pi$

THEOREM 7. *If $f(z)$ has the property $K(\rho)$, then the operator range of $f(A)$ covers some ball of radius*

$$k_1(\rho) = \sqrt{3}/(4N_1(\rho)), \quad \text{where } 0 < \rho < \pi.$$

Note that the estimates in Theorems 3 and 6 behave in two extreme cases. When ρ is small the estimate in Theorem 3 is better than that of Theorem 6, but the later is better than the former when ρ tends to π.

Furthermore, instead of the property $K(\rho)$, we can also consider extensions as suggested by a result of Doob [**2**, Theorem 10.2] and the author [**10**, Theorem 7]. For this, we denote by $D_0(r, \rho, s)$, the class of all functions $f(z)$ analytic in U such that $|f(0)| \leq r < 1$ and for some arc γ on the unit circle of measure $|\gamma| \geq 2\rho$, the inequality

$$\liminf_{z \to p} |f(z)| \geq s > 0 \qquad (z \in U)$$

holds for all points $p \in \gamma$ less a subset of zero capacity.

THEOREM 8. *If $f \in D_0(r, \rho, s)$, then the operator range of $f(A)$ covers some ball of radius*

$$k_2(\rho) = \sqrt{3}/(4N_2(\rho)), \qquad N_2(\rho) = s(1 - r)/\log(1/(1 - \cos\rho)),$$

$0 < \rho < \rho_0$ *for some* $\rho_0 < \pi/2$.

6. Problem. In closing this note, let us pose two problems as follows:

Problem 1. What is the best estimate of the radius $k(\rho)$ in Theorem 4?

In view of Theorem 2, the constant $\sqrt{3}/4$ in Theorem 4 should be improved by $1/2$, or even by $\pi/4$ (cf. [**10**, Remark 3]).

Problem 2. What is a necessary and sufficient condition that the operator range be convex for a convex function?

In [**4**, Theorem 8], Fan proved that the operator range of an extreme point e_θ of the class of convex functions is convex, where $e_\theta(z) = z(1 - e^{i\theta}z)^{-1}$. Is it true that if the operator range of a function $f(z) \neq z$ is convex then the function f is an extreme point, i.e. $f = e_\theta$ for some $0 \leq \theta < 2\pi$? We do not even know whether the operator range of any odd univalent function is not convex? Our original intention was to prove this stronger result. But it was pointed out by the referee that our method can only give the weaker version described in Theorem 5. The referee asked if the specific odd function $f(x) = z + z^3/3$ has nonconvex operator range. The answer turns out to be 'yes' as will be seen from the following geometric criterion:

THEOREM 9. *If $f(z)$ is an odd function whose range contains two points w_1 and w_2 such that $|w_1| = |w_2| > 1$ and $\arg(w_1/w_2) = \pi/2$, then $f(z)$ has nonconvex operator range.*

PROOF. According to the hypotheses, there exist two points ζ and η in U such that $f(\zeta) = if(\eta)$ and $|f(\zeta)| > 1$. Let $A_1 = \zeta\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $A_2 = i\eta\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then by a simple computation, we obtain

$$\frac{1}{2}(f(A_1) + f(A_2)) = f(\zeta)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The assertion now follows from the same argument as in Theorem 5.

As a consequence of Theorem 9, we have the following desired result:

THEOREM 10. *If $f_r(z) = z + rz^3$, $0 < r \leqslant \frac{1}{3}$, then $f_r(z)$ has nonconvex operator range.*

PROOF. Since

$$|f(e^{i\theta})| = 1 + r^2 + 2r\cos 2\theta > 1 \quad \text{for all } |\theta| \leqslant \pi/4,$$

there exists a unique $0 < \alpha < \pi/4$ such that $\arg f(e^{\pm i\alpha}) = \pm\pi/4$. The assertion now follows from Theorem 9.

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