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SYMMETRIC CONTINUITY OF REAL FUNCTIONS

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ABSTRACT. It is shown that the set of points where a real function is both symmetrically continuous and not continuous has inner measure zero but may have full outer measure.

We shall consider an arbitrary function f that maps the real line $\mathbf{R}$ into itself. Such a function is said to be *symmetrically continuous at* $x \in \mathbf{R}$ if

$$f(x+h) - f(x-h) \to 0 \quad \text{as } h \to 0.$$

The set of all points where f is symmetrically continuous is denoted by $\mathcal{SC}$, and the set of all points where f is discontinuous is denoted by $\mathfrak{D}$. (Recall that $\mathfrak{D}$ is of type F_σ; on the other hand, $\mathcal{SC}$ need not be measurable and need not possess the Baire property—examples are given below.)

The object of this investigation is the set $\mathcal{SC} \cap \mathfrak{D}$. Interest in this set apparently dates back to 1935 when F. Hausdorff [3] asked whether $\mathfrak{D}$ could be uncountable when $\mathcal{SC} = \mathbf{R}$; an affirmative answer to this question was given in 1971 by D. Preiss [4]. Hausdorff also asked whether an arbitrarily prescribed F_σ-set could equal $\mathfrak{D}$ when $\mathcal{SC} = \mathbf{R}$. In response to this question, H. Fried [2] proved the following result.

THEOREM F. *If $\mathcal{SC}$ is residual, then $\mathfrak{D}$ is of first category.*

The analogue for f restricted to an interval is clearly valid; and because $\mathfrak{D}$ has the Baire property and any residual subset of an open interval has the Baire property, this theorem has the following equivalent formulations.

THEOREM F*. *$\mathcal{SC} \cap \mathfrak{D}$ is residual on no open interval; or, equivalently, $\mathcal{SC} \cap \mathfrak{D}$ contains no set of second category that has the Baire property.*

In the following sense, Theorem F* is the "best" statement that can be made concerning the category of $\mathcal{SC} \cap \mathfrak{D}$. P. Erdös [1] has shown that the continuum hypothesis implies the existence of an additive subgroup G of $\mathbf{R}$ that is of measure 0 and of second category on every open interval. Thus, if f is the characteristic function of G, then $G \subset \mathcal{SC}$ and $\mathfrak{D} = \mathbf{R}$; that is, $\mathcal{SC} \cap \mathfrak{D}$ is of second category on every open interval.

Erdös also proved that the continuum hypothesis implies the existence of a subgroup G of $\mathbf{R}$ that is of first category and has full exterior measure on each open

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interval. Thus, if f is the characteristic function of G, then

$$|(\mathcal{SC} \cap \mathcal{D}) \cap I|_e = |I|$$

for each open interval I (where $|\cdot|_e$ and $|\cdot|$ denote exterior Lebesgue measure and Lebesgue measure). Consequently, the following metric analogue of Theorem F* is the "best" statement that can be made concerning the measure of $\mathcal{SC} \cap \mathcal{D}$. (Here $|\cdot|_i$ denotes inner Lebesgue measure.)

THEOREM. $|\mathcal{SC} \cap \mathcal{D}|_i = 0$.

It is our goal to furnish a proof of this theorem. First we note an immediate corollary: *if $\mathcal{SC}$ is measurable, then* $|\mathcal{SC} \cap \mathcal{D}| = 0$; *in particular, if $\mathcal{SC}$ is of full measure, then* $|\mathcal{D}| = 0$. This contains a previous result of Preiss **[4]**, who proved that $\mathcal{SC} = \mathbf{R}$ implies $|\mathcal{D}| = 0$. We also note that E. M. Stein and A. Zygmund **[5**, Lemma 9] presented our theorem with the added hypothesis that f be measurable. However, their proof contains an oversight; for example, if $f(x) = x$, then in their notation $E = \mathbf{R}$ and $E_n = \varnothing$ for each n. (In a footnote **[5**, p. 266], they do allude to a second proof which may be correct.)

To begin the proof, we introduce an auxiliary function of two variables and prove two key lemmas concerning it. For each $x \in \mathbf{R}$ and each $\varepsilon > 0$ we define $\delta(x; \varepsilon)$ to be the supremum of all numbers $\delta \geqslant 0$ for which

$$|f(x+h) - f(x-h)| < \varepsilon \quad \text{whenever } |h| \leqslant \delta.$$

(Note that $\delta(x; \varepsilon) \geqslant 0$, and that $\delta(x; \varepsilon) > 0$ for each $\varepsilon > 0$ if and only if $x \in \mathcal{SC}$.)

LEMMA 1. *Let ε and δ be positive numbers, and let I be an open interval with $|I| < \delta/2$. If $|\{x: \delta(x; \varepsilon) > \delta\} \cap I|_e > 0$, then there exists a positive number τ such that $\delta(x; 5\varepsilon) > \tau$ for each $x \in \mathcal{SC} \cap I$.*

PROOF. Set $E = \{x: \delta(x; \varepsilon) > \delta\} \cap I$. Since $|E|_e > 0$, the distance set

$$d(E) \equiv \{|x-y| : x, y \in E\}$$

is dense on some interval $(0, \tilde{\tau})$. (This follows from the identity $\overline{d(E)} = d(\bar{E})$ and the well-known fact that the distance set of a set of positive measure contains some interval $(0, \tilde{\tau})$.) Choose any point $x \in \mathcal{SC} \cap I$, and without loss of generality assume $x = 0$. Let $h \in (0, \tilde{\tau})$ be arbitrary. To establish the lemma for any $\tau \in (0, \tilde{\tau})$, we shall show that

$$(1) \qquad |f(h) - f(-h)| < 5\varepsilon.$$

Inequality (1) is certainly satisfied if $\delta(0; \varepsilon) > h$. Suppose $\delta(0; \varepsilon) \leqslant h$. Because $d(E)$ is dense on $(0, \tilde{\tau})$, there exist two points $x, y \in E$ such that $x < y$ that

$$(2) \qquad 0 < h - 2(y-x) < \delta(0; \varepsilon).$$

Let h_y be the reflection of h in y, and let h_{yx} be the reflection of h_y in x. Since $\tau \leqslant |I| < \delta/2$, it readily follows that

$$|h - y| < |I| + \tau < \delta \quad \text{and} \quad |h_y - x| < |I| + \tau < \delta.$$

Thus, since $x, y \in E$ we have

$$|f(h) - f(h_y)| < \varepsilon \quad \text{and} \quad |f(h_y) - f(h_{yx})| < \varepsilon;$$

that is,

$$(3) \qquad |f(h) - f(h_{yx})| < 2\varepsilon.$$

A similar argument shows that

$$(4) \qquad |f(-h) - f((-h)_{xy})| < 2\varepsilon,$$

where $(-h)_{xy}$ is the reflection in y of the reflection of $-h$ in x. Furthermore,

$$(-h)_{xy} = -h + 2(y - x) = -h_{yx},$$

and by (2) we have $0 < h_{yx} < \delta(0; \varepsilon)$; therefore,

$$|f(h_{yx}) - f((-h)_{xy})| < \varepsilon.$$

This together with (3) and (4) and the triangle inequality gives (1), and the proof is complete.

LEMMA 2. *Let ε and δ be positive numbers, and let I be an open interval with $|I| < \delta$. If $|\{x: \delta(x; \varepsilon) > \delta\} \cap I|_i > 3|I|/4$, then $|f(a) - f(b)| < 2\varepsilon$ for any two points $a, b \in I$.*

PROOF. If $E = \{x: \delta(x; \varepsilon) > \delta\} \cap I$, then it readily follows from the hypothesis on E that $(0, |I|/2) \subset d(E)$. Thus if $a, b \in I$ with $a < b$, there exist points $x, y \in E$ such that $x < y$ and $b - a = 2(y - x)$. Let a_x denote the reflection of a in x, and let a_{xy} denote the reflection of a_x in y. Because $|I| < \delta$ and $a_{xy} = b$ we have $|a - x| < \delta$ and $|a_x - y| < \delta$; and because $x, y \in E$ we have

$$|f(a) - f(a_x)| < \varepsilon \quad \text{and} \quad |f(a_x) - f(a_{xy})| < \varepsilon.$$

As $a_{xy} = b$, this implies that $|f(a) - f(b)| < 2\varepsilon$, and the lemma is established.

PROOF OF THEOREM. Assume $|\mathcal{SC} \cap \mathcal{D}|_i > 0$. If $\mathcal{D}_t$ denotes the set of points where the oscillation of f exceeds t, then $|\mathcal{SC} \cap \mathcal{D}_t|_i > 0$ for some $t > 0$ (since each $\mathcal{D}_t$ is measurable). Let M be a measurable subset of $\mathcal{SC} \cap \mathcal{D}_t$ with $|M| > 0$. Choose any $\varepsilon \in (0, t/11)$. Since $M \subset \mathcal{SC}$, there exists a $\delta > 0$ such that

$$|\{x: \delta(x, \varepsilon) > \delta\} \cap M|_e > 0.$$

Therefore, there exists a point $p \in \{x: \delta(x; \varepsilon) > \delta\} \cap M$ that is a point of density of M and a point of exterior density of the set $\{x: \delta(x; \varepsilon) \geq \delta\}$. Let I be an open interval containing p with $|I| < \delta/2$. By Lemma 1 there exists a $\tau > 0$ such that $\delta(x; 5\varepsilon) > \tau$ for each $x \in M$. Let J be an open interval containing p such that $|J| < \tau$ and $|M \cap J| > 3|J|/4$. Then

$$|\{x: \delta(x; 5\varepsilon) > \tau\} \cap J|_i > 3|J|/4,$$

and by Lemma 2 the oscillation of f on J does not exceed 10ε which is less than $10t/11$. This contradicts $p \in \mathcal{D}_t$, and the theorem is proved.

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