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A CONVERSE TO THE LUSIN-PRIVALOV RADIAL UNIQUENESS THEOREM

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ABSTRACT. Let E be a subset of the unit circumference C. If for every nonempty open arc A of C, the set E is not both metrically dense and of second category in A, then there exists a nonconstant analytic function f on the open unit disk Δ, such that $f^*(\eta) = 0$, $\eta \in E$, where f^* is the radial limit function of f.

Let $\Delta = \{|z| < 1\}$ and let E be a subset of $C = \{|z| = 1\}$. For f an analytic function on Δ, denote by f^* the radial limit function of f. Thus $f^*(\eta) = \lim_{r \to 1} f(r\eta)$ for each η in C where the limit exists (finite or infinite). According to a classical theorem of Fatou [**2**], a bounded analytic function f on Δ has radial limits almost everywhere in C. F. and M. Riesz [**6**] proved that if there exists a nonconstant bounded analytic function f on Δ such that $f^*(\eta) = 0$, $\eta \in E$, then E is of measure 0. A converse to the Riesz theorem was provided by Privalov [**5**, p. 214].

THEOREM 1. *If E is of measure* 0, *then there exists a nonconstant bounded analytic function f on Δ such that $f^*(\eta) = 0$, $\eta \in E$.*

Here we are concerned with a converse to the following celebrated uniqueness theorem of Lusin and Privalov [**3**, pp. 187–189].

THEOREM 2. *If there exists a nonconstant analytic function f on Δ such that $f^*(\eta) = 0$, $\eta \in E$, then for every nonempty open arc A in C, the set E is not both metrically dense and of second category in A.*

By definition E is metrically dense in an open arc A if, for every nonempty open subarc B of A, the set $E \cap B$ has positive outer measure. Also, E is nowhere dense if int $\bar{E} = \varnothing$, that is, the interior of the closure of E is empty, and is of first (resp. second) category if it is (resp. is not) a countable union of nowhere dense sets.

Lusin and Privalov [**3**, §§11 and 33–35] constructed nonconstant analytic functions f and g such that $f^*(\eta) = 0$ for a set of η in C of measure 2π and $g^*(\eta) = 0$ for a set of η in C which is of measure π in the upper half-circle and of second category in the lower half-circle. These examples show that some nontrivial combination of the second category and metric density conditions in Theorem 2 is indeed necessary.

In this paper, the full converse of Theorem 2 is proved.

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THEOREM 3. *If, for every nonempty open arc A in C, the set E is not both metrically dense and of second category in A, then there exists a nonconstant analytic function f on Δ such that $f^*(\eta) = 0$, $\eta \in E$.*

We shall need the following decomposition lemma.

LEMMA 1. *Suppose that for every nonempty open arc A in C, the set E is not both metrically dense and of second category in A. Then there exists a closed subset F of C such that $E \cap F$ is of first category and $E \setminus F$ is of measure* 0.

PROOF. Let F be the set of η in C such that for every open arc A containing η, the set $E \cap A$ has positive outer measure. Evidently F is closed.

$E \cap F$ is of first category. Since $F = (F \setminus \operatorname{int} F) \cup \operatorname{int} F$ and $F \setminus \operatorname{int} F$ is nowhere dense, it suffices to check that $E \cap \operatorname{int} F$ is of first category. Putting aside the trivial case, assume that $\operatorname{int} F \neq \varnothing$. Let A be a component (open arc) of int F. We claim that E is metrically dense in A. If not, there exists some nonempty open subarc B of A such that $E \cap B$ has measure 0. But this contradicts the fact that $B \subset F$ and the definition of F, so the claim is verified. Since, by assumption, E is not both metrically dense and of second category in A, we conclude that $E \cap A$ is of first category. The required conclusion now follows from the fact that int F has at most countably many components.

$E \setminus F$ is of measure 0. If $F = C$ the assertion is trivial, so assume $C \setminus F \neq \varnothing$. By the definition of F, there exists for each η in $C \setminus F$ an open arc A_η containing η such that $E \cap A_\eta$ has zero measure. By the countable basis property of C, the open cover $(A_\eta)_{\eta \in C \setminus F}$ of $C \setminus F$ has a countable subcover $(A_{\eta_k})_1^\infty$, where $\{\eta_k\}_1^\infty \subseteq C \setminus F$. Now $E \cap A_{\eta_k}$ has zero measure for each k, so the same follows for $E \setminus F$ since $E \setminus F \subseteq \bigcup_1^\infty (E \cap A_{\eta_k})$.

Lemma 1 is established.

We shall also require a slightly strengthened form of a theorem of Bagemihl and Seidel [**1**].

THEOREM 4. *If E is of first category, then there exists a nonconstant analytic function f on Δ which is analytic at each point of $C \setminus \bar{E}$ and satisfies $f^*(\eta) = 0$, $\eta \in E$.*

The proof of Theorem 4 that we shall give, except for the modification used here to guarantee that f is analytic in $C \setminus \bar{E}$, follows the outline given by Schneider in [**7**, p. 335].

PROOF. If $E = \varnothing$, let $f(z) = \exp(z)$, $z \in \Delta$. Assume now that $E \neq \varnothing$. Let R be the region $\Delta \cup \{0 < |z| < 2,\ z/|z| \in C \setminus \bar{E}\}$ and $(F_n)_1^\infty$ a monotone nondescending sequence of closed nowhere dense sets such that $E \subseteq \bigcup_1^\infty F_n \subseteq \bar{E}$. Define $W_n = \{\eta \in C: \operatorname{dist}(\eta, \bar{E}) \geqslant 1/n\}$ for each positive integer n, where

$$\operatorname{dist}(z, W) = \inf_{w \in W} |z - w| \quad \text{for } z \in \mathbf{C},\ W \subseteq \mathbf{C},$$

and note that $(W_n)_1^\infty$ is a monotone nondescending sequence of compact subsets of $C \setminus \bar{E}$ such that $\bigcup_1^\infty \operatorname{int} W_n = C \setminus \bar{E}$. Let $S_n = \{|z| \leq 1 - 1/n\} \cup \{0 < |z| \leq 2, z/|z| \in W_n\}$ and $T_n = \{1 - 1/n \leq |z| \leq 1, z/|z| \in F_n \text{ when } z \neq 0\}$, and define

$$(1) \qquad h_n(z) = \begin{cases} 0, & z \in S_n, \\ n2^n[|z| - (1 - 1/n)], & z \in T_n, \end{cases}$$

for all positive integers n.

Since h_n is continuous on the compact set $K_n = S_n \cup T_n$ and analytic in its interior, and $C \setminus K_n$ is connected, Mergelyan's Theorem [4] implies the existence of a polynomial p_n such that

$$(2) \qquad |p_n(z) - h_n(z)| < 1/2^n, \qquad z \in K_n,$$

for every positive integer n. Noting that for each compact subset K of the region R there exists an n such that $K \subseteq S_n$, and that this implies $h_n(z) = 0$, $z \in K$, by (1), we see from (2) and the Weierstrass M-test that $\sum_1^\infty p_n$ converges uniformly on compact subsets of R to a function g analytic on R.

Let f be the restriction of $\exp(-g)$ to Δ. Then f is a nonvanishing analytic function on Δ which is analytic at each point of $C \setminus \bar{E}$. We claim that $f^*(\eta) = 0$, $\eta \in E$. It suffices to show that $\lim_{r \to 1} \operatorname{Re} g(r\eta) = +\infty$, $\eta \in E$. Let $\eta \in E$. Then there exists a positive integer n such that $\eta \in F_j$ for $j \geq n$. If $m \geq n$, it follows from (1) and (2) that

$$(3) \qquad \begin{aligned} \operatorname{Re} g(r\eta) &\geq 2^{m-1} - \sum_{\substack{j \geq n \\ j \neq m}} 2^{-j} - \sum_1^{n-1} |p(r\eta)| \\ &\geq 2^{m-1} - \alpha, \qquad 1 - 1/(2m) \leq r < 1, \end{aligned}$$

where α is a positive constant not depending on m. The required conclusion follows and the claim is verified. Finally, f is nonconstant since it is nonvanishing and $f^*(\eta) = 0$, $\eta \in E \neq \varnothing$. The proof is completed.

We are now ready to prove Theorem 3.

PROOF OF THEOREM 3. Assume $E \neq \varnothing$. Take F as in Lemma 1. Since $E \setminus F$ has zero measure, there exists, by Theorem 1, a nonconstant bounded analytic function g on Δ such that $g^*(\eta) = 0$, $\eta \in E \setminus F$. Since $E \cap F$ is of first category, Theorem 4 implies there exists a nonconstant analytic function h on Δ such that h is analytic at each point of $C \setminus (\overline{E \cap F}) \supseteq C \setminus F$ and $h^*(\eta) = 0$, $\eta \in E \cap F$.

Let $f = gh$. Then f is analytic on Δ. From the condition $g^*(\eta) = 0$, $\eta \in E \setminus F$, and the analyticity of h at each point of $C \setminus F$, we have $f^*(\eta) = 0$, $\eta \in E \setminus F$. On the other hand, since $h^*(\eta) = 0$, $\eta \in E \cap F$, and g is bounded, we conclude that $f^*(\eta) = 0$, $\eta \in E \cap F$. Thus $f^*(\eta) = 0$, $\eta \in (E \setminus F) \cup (E \cap F) = E$. The nonconstancy of f follows from the fact that it is a product of nonconstant functions and that $f^*(\eta) = 0$, $\eta \in E \neq \varnothing$. This completes the proof.

We note that since the functions constructed by Privalov for Theorem 1 and here for Theorem 4 are nonvanishing, the function f of Theorem 3 may also be required to be nonvanishing.

REFERENCES

1. F. Bagemihl and W. Seidel, *Some boundary properties of analytic functions*, Math. Z. **61** (1954), 186–199.

2. P. Fatou, *Séries trigonométriques et séries de Taylor*, Acta Math. **30** (1906), 335–400.

3. N. N. Lusin and I. I. Privalov, *Sur l'unicité et la multiplicité des fonctions analytiques*, Ann. Sci. École Norm. Sup. (3) **42** (1925), 143–191.

4. S. N. Mergelyan, Amer. Math. Soc. Transl. **3** (1962), 287–293.

5. I. I. Privalov, *Randeigenschaften analytischer Funktionen*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1956.

6. F. Riesz and M. Riesz, *Über die Randwerte einer analytischen Funktion*, Quatrième Congrès des Math. Scand. Stockholm, 1916, pp. 27–44.

7. W. J. Schneider, *Approximation and harmonic measure*, Aspects of Contemporary Complex Analysis (D. H. Brannan and J. G. Clunie, eds.), Academic Press, New York, 1980, pp. 334–340.

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