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CLOSABLE OPERATORS AND SEMIGROUPS

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ABSTRACT. We show that a linear operator is closable iff it is possible to weaken the topology on its range in a certain nice way so as to render the operator continuous. We apply this result to show that if E is a sequentially complete locally convex Hausdorff space and $(L(t))_{0 \leq t < \infty}$ is a locally equicontinuous semigroup of class (C_0) in E with generator S and if $x \in E$ (not necessarily belonging to the domain of S) then the function $u(t) = L(t)x$ is a solution, in a generalized sense, of the initial value problem $u'(t) = Su(t)$, $u(0) = x$, and that such a generalized solution is unique. It should be noted here that $u(t)$ may fail to belong to the domain of S so we must assign a suitable meaning to the expression $Su(t)$.

THEOREM 1. *Let E and F be topological vector spaces and let S be a linear operator from E into F with domain $D \subseteq E$. Then S is closable iff there is a Hausdorff linear topology τ on F which is weaker than the original topology on F and which renders S continuous. In this case, there is a canonical description of the strongest such topology τ.*

PROOF. As is usual, we identify S with its graph. Define ψ: $D \times F \to F$ by $\psi(x, y) = y - Sx$. Then ψ is a linear map, the kernel of ψ is S, and the range of ψ is all of F. Let τ be the quotient topology induced on F by ψ. Then it is easy to check that τ is the strongest linear topology on F which is weaker than the original topology of F and which renders S continuous. Thus we need only show that S is closable (as an operator from E to F) iff τ is Hausdorff. But, as is well known, S is closable iff S is closed in $D \times F$, and τ is Hausdorff iff the kernel of ψ, namely S, is closed in $D \times F$. This completes the proof.

THEOREM 2. *Let E, F, D, and S be as in Theorem 1. Assume D is dense in E. Suppose S is closable and let τ be the strongest linear topology on F which is weaker than the original topology of F and which renders S continuous. Let G be the completion of (F, τ), let T be the unique extension of S to a continuous linear map of E into G, and let $\bar{S}$ be the closure of S in $E \times F$. Then $T \supseteq \bar{S}$, and if $x \in E$ and $Tx \in F$ then x is an element of the domain of $\bar{S}$.*

PROOF. It is clear that $T \supseteq \bar{S}$. Suppose $x \in E$ and $y = Tx \in F$. Let (x_i) be a net in D which converges in E to x. Let U and V be neighbourhoods of 0 in E and F respectively. Let $W = \psi[(U \cap D) \times V]$ where ψ is as in the proof of Theorem 1. As ψ is linear and τ is the quotient topology induced on F by ψ, ψ is open onto (F, τ) so W is a neighbourhood of 0 in (F, τ). Now $(y - Sx_i)$ τ-converges to 0. Thus there exists i_0 such that $i \geq i_0$ implies $\psi(x_i, y) \in W$; i.e., $(x_i, y) \in [(U \cap D) \times V] + S$.

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Hence $(x, y) \in \overline{(U \times V) + S}$. But $\bar{S}$ is the intersection of the sets of the form $\overline{(U \times V) + S}$ where U and V range over neighbourhoods of 0 in E and F respectively. Thus $(x, y) \in \bar{S}$. Hence x is an element of the domain of $\bar{S}$. This completes the proof.

Now let E be a sequentially complete locally convex Hausdorff topological vector space, which will be fixed throughout the rest of the discussion. We shall give some applications of Theorems 1 and 2 to the study of semigroups of operators on E. Let $(L(t))_{0 \leq t < \infty}$ be a locally equicontinuous semigroup of class (C_0) on E. We recall that this means:

(a) Each $L(t)$ is a continuous linear operator on E.

(b) $L(0) = I$ and $L(s + t) = L(s)L(t)$ for all s and t.

(c) For each $x \in E$, the map $t \mapsto L(t)x$ is continuous on $[0, \infty)$.

(d) For any $a \in [0, \infty)$, $\{L(t)\colon 0 \leq t \leq a\}$ is equicontinuous.

Let S be the generator of $(L(t))$. Then S is a closed, densely defined, linear operator in E. (The proof given in [**1**] for this standard result in the case where E is a Banach space can easily be generalized to the present setting.) Let D be the domain of S and let τ be the strongest linear topology on E which is weaker than the original topology on E and which renders S continuous from D with its original topology to (E, τ). By Theorem 1, τ is Hausdorff because S is closed. Also, by the description of τ given in the proof of Theorem 1, τ is locally convex. Let F be the completion of (E, τ) (then F is also locally convex) and let T be the unique extension of S to a continuous linear map of E into F.

It is natural to ask whether $(L(t))$ can be extended to a semigroup on F and what properties this extended semigroup may have. The next result deals with this question.

THEOREM 3. *The following statements hold*:

(a) *For each* $t \in [0, \infty)$, *there is a unique continuous linear operator* $M(t)$ *on* F *whose restriction to* E *is* $L(t)$.

(b) $(M(t))$ *is a locally equicontinuous semigroup of class* (C_0) *on* F *and the generator* A *of* $(M(t))$ *is an extension of* T.

(c) *If* E *is complete and if for some real number* β, $\{e^{-t\beta}L(t)\colon 0 \leq t < \infty\}$ *is equicontinuous on* E, *then* $A = T$.

(We remark that if E is a Banach space then the hypotheses of (c) are automatically satisfied; see [**3**, p. 232].)

PROOF. Let ψ be the map $(x, y) \mapsto y - Sx$ from $D \times E$ onto E. It is a well-known observation that if $x \in D$ then for $0 \leq t < \infty$ we have $L(t)x \in D$ and $SL(t)x = L(t)Sx$. From this it follows that $L(t)\psi(x, y) = \psi(L(t)x, L(t)y)$. Suppose $a \in (0, \infty)$ and W is any neighbourhood of 0 in (E, τ). Then there are neighbourhoods U and V of 0 in E such that $\psi[(D \cap V) \times V] \subseteq W$ and $L(t)[U] \subseteq V$ for $0 \leq t \leq a$. But then $L(t)[\psi[(D \cap U) \times U]] \subseteq W$ for $0 \leq t \leq a$. Now $\psi[(D \cap U) \times U]$ is a neighbourhood of 0 in (E, τ) since ψ is open onto (E, τ). Thus $\{L(t)\colon 0 \leq t \leq a\}$ is an equicontinuous family of linear operators on (E, τ). Part (a) follows from this. Moreover, since the closure in F of any neighbourhood of 0 in (E, τ) is a neighbourhood of 0 in F, it follows that $\{M(t): 0 \leqslant t \leqslant a\}$ is equicontinuous on F for any $a \in (0, \infty)$. (A similar argument shows that under the hypotheses of (c), $\{e^{-t\beta}M(t): 0 \leqslant t < \infty\}$ will be equicontinuous on F.) That $M(0) = I$ and $M(s + t) = M(s)M(t)$ is clear. The continuity of $t \mapsto M(t)x$ for $x \in F$ is proved by a uniform convergence argument using the local equicontinuity of $(M(t))$. It is obvious that $A \supseteq S$. Suppose $x \in E$ and $y = Tx$. Then there is a net (x_i) in D converging to x in E. Then $Sx_i \to y$ in F and $x_i \to x$ in F. As $A \supseteq S$ and A is closed in $F \times F$, x belongs to the domain of A and $Ax = y$. Thus $A \supseteq T$.

Now suppose the hypotheses of part (c) are satisfied. Let $\lambda \in (\beta, \infty)$. Then $(\lambda - S)^{-1}$ exists and is continuous from E into E and $R := (\lambda - A)^{-1}$ exists and is continuous from F into F; see [**3**, p. 240]. Since $A \supseteq S$, $R \supseteq (\lambda - S)^{-1}$. It is easy to check that $(\lambda - S)^{-1} \circ \psi$ is continuous from $D \times E$ into E. It follows that $(\lambda - S)^{-1}$ is continuous from (E, τ) into E. Therefore the range of R is contained in E, since E is complete. Hence $A = T$. This completes the proof.

If $x \in D$ then $u(t) = L(t)x$ is the unique solution of the initial value problem $u'(t) = Su(t)$, $u(0) = x$. As the next result shows, this is actually true for any $x \in E$ and not just for $x \in D$, provided we replace S by its extension T and compute the derivative $u'(t)$ relative to the topology of F instead of E.

THEOREM 4. *The following statements hold*:

(a) *For each* $x \in E$, *we have* $F - dL(t)x/dt = TL(t)x$ *on* $[0, \infty)$.

(b) *If* $a \in (0, \infty)$, $x \in E$, *and* u: $[0, a] \to E$ *is continuous from* $[0, a]$ *to* F *and satisfies* $F - du(t)/dt = Tu(t)$ *on* $(0, a)$ *and* $u(0) = x$ *then* $u(t) = L(t)x$ *for* $0 \leqslant t \leqslant a$.

PROOF. Let $(M(t))$ and A be as in Theorem 3. Then $L(t)x = M(t)x$ and $TL(t)x = AL(t)x$. Thus (a) just says $F - dM(t)x/dt = AM(t)x$ which, as is well known, holds for all x in the domain of A and in particular for all $x \in E$.

To prove (b), consider any $b \in (0, a]$. Define y: $[0, b] \to E$ by $y(t) = L(b - t)u(t) = M(b - t)u(t)$. Using the equicontinuity of $\{M(s): 0 \leqslant s \leqslant b\}$ on F, one can easily check that y is continuous from $[0, b]$ into F and $F - dy(t)/dt = 0$ on $(0, b)$. Hence $y(b) = y(0)$; i.e., $u(b) = L(b)u(0) = L(b)x$. This completes the proof.

As a final application we give an alternative proof of a result of K. Yosida [**2**, p. 58], to the effect that the weak generator of $(L(t))$ is the same as S.

THEOREM 5. *Suppose* x, $y \in E$ *and*

$$\operatorname{w-lim}_{h \downarrow 0} \frac{L(h) - I}{h} x = y,$$

where w-lim *denotes weak limit in* E. *Then* $x \in D$ *and* $y = Sx$.

PROOF. The hypotheses implies that

$$F\text{-}\operatorname{w-lim}_{h \downarrow 0} \frac{M(h) - I}{h} x = y$$

where M is as in Theorem 3 and F-w-lim denotes weak limit in F. But by (b) of Theorem 3,

$$F\text{-}\lim_{h\downarrow 0}\frac{M(h)-I}{h}x = Tx.$$

Since F is locally convex and Hausdorff, its weak topology is also Hausdorff. Hence $y = Tx$. Since S is closed and $Tx \in E$, we have $x \in D$ by Theorem 2. Hence $y = Sx$. This completes the proof.

References

1. W. Rudin, *Functional analysis*, McGraw-Hill, New York, 1973.

2. K. Yosida, *Lectures on semi-group theory and its application to Cauchy's problem in partial differential equations*, Tata Institute of Fundamental Research, Bombay, 1957.

3. ______, *Functional analysis*, 5th ed., Springer-Verlag, Berlin, Heidelberg and New York, 1978.

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