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A NOTE ON NEIGHBOURHOODS OF UNIVALENT FUNCTIONS

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ABSTRACT. Using a notion of neighbourhood of analytic functions due to Stephan Ruscheweyh we examine conditions under which neighbourhoods of a certain class of convex functions are included in a class of starlike functions.

Introduction. Let A denote the class of analytic functions f in the unit disk E: $\{z \mid |z| < 1\}$ with $f(0) = 0$, $f'(0) = 1$. For $f(z)$: $z + \sum_{k=2}^{\infty} a_k z^k \in A$ and $\delta \geq 0$ Ruscheweyh has defined the neighbourhood $N_\delta(f)$ as follows:

$$N_\delta(f) := \left\{ g(z) := z + \sum_{k=2}^{\infty} b_k z^k \;\middle|\; \sum_{k=2}^{\infty} k\,|a_k - b_k| \leq \delta \right\}.$$

He has shown in [1] among other results that if $f(z) := z + \sum_{k=n+1}^{\infty} a_k z^k \in C$ the following result is true:

$$N_{\delta_n}(f) \subset S^*, \qquad \delta_n = 2^{-2/n},$$

where $C(S^*)$ denotes the class of normalized convex (starlike) univalent functions in A. He also asked if a similar result would hold if we replace S^* by the class

$$T := \left\{ g \in S^* \;\middle|\; \left| z\frac{g'(z)}{g(z)} - 1 \right| < 1, z \in E \right\}$$

and C by the class

$$\tilde{T} := \left\{ g \in C \;\middle|\; \left| z\frac{g''(z)}{g'(z)} \right| < 1, z \in E \right\}.$$

We prove

THEOREM 1. *Let* $f(z) := z + \sum_{k=n+1}^{\infty} a_k z^k \in \tilde{T}$. *Then* $N_{\delta_n}(f) \subset T$, $\delta_n = e^{-1/n}$.

Let S^*_α $(0 \leq \alpha < 1)$ denotes the class $\{g \in S^* \mid \mathrm{Re}[z(g'(z)/g(z))] > \alpha,\ z \in E\}$. An analogue of this class with respect to T is the class

$$T_r := \left\{ g \in T \;\middle|\; \left| z\frac{g'(z)}{g(z)} - 1 \right| < r, z \in E \right\}, \qquad 0 < r \leq 1.$$

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In [1] Ruscheweyh has shown that for no $\alpha \in [0, 1)$ is there a positive δ such that $N_\delta(S_\alpha^*) \subset S^*$. For the class T_r the situation is quite different as shown by

THEOREM 2. *Let* $g(z) := z + \sum_{k=n+1}^{\infty} a_k z^k \in T_r$, $0 \leq r < 1$. *Then* $N_{\delta_n}(g) \subset T$, $\delta_n = e^{-r/n}(1 - r)$.

The boundaries of $\{w \in \hat{\mathbf{C}} \mid \mathrm{Re}[w] > 0\}$ and of $\{w \in \hat{\mathbf{C}} \mid \mathrm{Re}[w] > \alpha\}$ are not disjoint whereas those of $\{w \in \hat{\mathbf{C}} \mid |w - 1| < 1\}$ and of $\{w \in \hat{\mathbf{C}} \mid |w - 1| < r\}$ are; this is one of the reasons for the difference between the two situations. Nevertheless Theorem 2 is still interesting since the value for δ_n is best possible.

Concerning this question of boundaries we can prove

THEOREM 3. *Let* $f \in T$ *and* $D := \{zf'(z)/f(z) \mid z \in E\}$ *be such that there is* $w \in \overline{D}$ *with* $|w - 1| = 1$. *Then for no* $\delta > 0$ *we have* $N_\delta(f) \subset T$.

It should be noted that no similar result holds if the class T is replaced by the class S^*; in fact for $f(z) = z/(1 - z) \in C \subset S^*$ we have $N_{1/4}(f) \subset S^*$ even though the region $D = \{w \in \mathbf{C} \mid \mathrm{Re}[w] > \frac{1}{2}\}$ is such that the point at infinity belongs to both $\overline{D}$ and $\{w \in \hat{\mathbf{C}} \mid \mathrm{Re}[w] \geq 0\}$.

Proof of Theorem 1. It was established in [2] that for $f(z) := z + \sum_{k=2}^{\infty} a_k z^k \in T$ we have the estimate $|z| e^{-|z|} \leq |f(z)| \leq |z| e^{|z|}$; using the same method it is very easy to show that for $f(z) := z + \sum_{k=n+1}^{\infty} a_k z^k \in T$ the estimate

$$|z| e^{-|z|^n/n} \leq |f(z)| \leq |z| e^{|z|^n/n} \tag{1}$$

is true and sharp as seen from the function $f(z) := ze^{z^n/n}$. We also remark that $f(z) \in \tilde{T} \Leftrightarrow zf'(z) \in T$ so that we obtain for $f(z) := z + \sum_{k=n+1}^{\infty} a_k z^k \in \tilde{T}$ for the following estimate

$$e^{-|z|^n/n} \leq |f'(z)| \leq e^{|z|^n/n} \tag{2}$$

and the sharpness is established by looking at the function $f(z) := \int_0^z e^{u^n/n}\, du$.

We also remark the following: a function $g(z) \in A$ belongs to the class T iff for every $\theta \in [0, 2\pi)$ we have

$$z\frac{g'(z)}{g(z)} - 1 \neq e^{i\theta}, \qquad z \in E,$$

that is

$$\frac{1}{z}\left(\left(\frac{z/(1 - z)^2 - (1 + e^{i\theta})z/(1 - z)}{-e^{i\theta}}\right) * g(z)\right) \neq 0, \qquad \theta \in [0, 2\pi), z \in E,$$

where $*$ denotes the Hadamard product of two functions. Since

$$-e^{i\theta}h_\theta(z) := \frac{z}{(1 - z)^2} - (1 + e^{i\theta})\frac{z}{1 - z} = -e^{i\theta}z + \sum_{n=2}^{\infty} (n - 1 - e^{i\theta})z^n$$

where $|n-1-e^{i\theta}|\leqslant n$ it is clear from the results in [1] that a sufficient condition in order that $N_\delta(f)\subset T$ may hold for some function f in A is that

$$\left|\frac{h_\theta(z)*f(z)}{z}\right|\geqslant\delta,\qquad z\in E,\theta\in[0,2\pi). \tag{3}$$

Now let $f(z):=z+\sum_{k=n+1}^{\infty}a_kz^k\in\tilde{T}$. We have

$$f(z)*h_\theta(z)=\frac{zf'(z)-(1+e^{i\theta})f(z)}{-e^{i\theta}},$$

$$\frac{(f(z)*h_\theta(z))'}{f'(z)}=1-e^{-i\theta}z\frac{f''(z)}{f'(z)}$$

with $\operatorname{Re}[1-e^{i\theta}zf''(z)/f'(z)]\geqslant 1-|zf''(z)/f'(z)|>0$. This shows, since $f\in\tilde{T}\subset C$, that the functions $h_\theta(z)*f(z)$ are close-to-convex univalent. We also get the estimate

$$|(f(z)*h_\theta(z))'|\geqslant|f'(z)|\left(1-\left|z\frac{f''(z)}{f'(z)}\right|\right)\geqslant e^{-|z|^n/n}(1-|z|^n)$$

using (2) and Schwarz lemma. Since the functions $h_\theta(z)*f(z)$ are univalent we can integrate the last estimate to obtain

$$|f(z)*h_\theta(z)|\geqslant\int_0^{|z|}e^{-u^n/n}(1-u^n)\,du=|z|e^{-|z|^n/n}$$

so that according to (3), $N_{\delta_n}(f)\subset T$ for $\delta_n=e^{-1/n}$. The sharpness of the result is seen from the function $f(z):=\int_0^z e^{u^n/n}\,du$; in fact $g(z):=f(z)+\delta_nz^{n+1}/(n+1)\in N_{\delta_n}(f)$ and $g'(z)=f'(z)+\delta_nz^n=0$ if $z^n=-1$. This completes the proof of Theorem 1.

Proof of Theorem 2. The proof of Theorem 2 is more direct. We first remark that from the definition of T_r we have

$$g(z)\in T_r\Leftrightarrow g(z)=z\left(\frac{g_1(z)}{z}\right)^r\quad\text{for some function } g_1\in T$$

so that if $g(z):=z+\sum_{k=n+1}^{\infty}a_kz^k\in T_r$ we get from (1) and Schwarz lemma that

$$e^{-r|z|^n/n}\leqslant\left|\frac{g(z)}{z}\right|\leqslant e^{r|z|^n/n}, \tag{4}$$

$$\left|z\frac{g'(z)}{g(z)}\right|\leqslant r|z|^n. \tag{5}$$

Now let $0\leqslant\theta<2\pi$; we have, according to (4) and (5), for $z\in E$,

$$\left|\frac{g(z)*h_\theta(z)}{z}\right|=\left|g'(z)-(1+e^{i\theta})\frac{g(z)}{z}\right|\geqslant\left|\frac{g(z)}{z}\right|\left(1-\left|z\frac{g'(z)}{g(z)}-1\right|\right)$$
$$\geqslant e^{-r|z|^n/n}(1-r|z|^n)$$

from which it follows, according to (3), that $N_{\delta_n}(g)\subset T$ for $\delta_n=(1-r)e^{-r/n}$. The sharpness of the result is seen from the function $g(z):=ze^{rz^n/n}$; in fact,

$f(z) := g(z) + \delta_n z^{n+1}/(n+1) \in N_{\delta_n}(g)$ and $f'(z) = 0$ if $z^n = -1$. This completes the proof of Theorem 2.

Proof of Theorem 3. Let $h_\theta(z)$ be defined as before. Since

$$\frac{f(z) * h_\theta(z)}{z} = -e^{i\theta}\frac{f(z)}{z}\left(\left(z\frac{f'(z)}{f(z)} - 1\right) - e^{i\theta}\right)$$

it is clear from the hypothesis on D, that, $|f(z)/z|$ being bounded in E,

$$\inf\left|\frac{f(z) * h_\theta(z)}{z}\right| = 0 \tag{6}$$

where the inf is taken over all $z \in E$, $\theta \in [0, 2\pi)$.

We now proceed to show Theorem 3 following an idea due to Ruscheweyh [**1**]. Let $\delta > 0$ and n some integer > 2. Choose a point $z_0 \in E$ and $\theta \in [0, 2\pi)$ such that for $\mu := (f * h_\theta(z_0))/z_0^n$ we have

$$|\mu| = \left|\frac{f * h_\theta(z_0)}{z_0^n}\right| < \delta\left(\frac{n-2}{n}\right).$$

This is always possible because of (6) and the fact that the function $f(z) * h_\theta(z)$, f being in the class T, is nonvanishing for $z \neq 0$. We then define the function $g(z) := f(z) - \mu z^n/a_n$ where $a_n := h_\theta^{(n)}(0)/n! = (n - 1 - e^{i\theta})/-e^{i\theta}$; it is clear that $|a_n| \geqslant n - 2$ so that $n|\mu/a_n| \leqslant n|\mu|/(n-2) < \delta$ and $g \in N_\delta(f)$; but on the other side we have

$$\frac{g * h_\theta(z_0)}{z_0} = \frac{f * h_\theta(z_0)}{z_0} - \mu z_0^{n-1} = 0$$

which shows that $g \notin T$. This completes the proof of Theorem 3.

References

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