A NOTE ON NEIGHBOURHOODS OF UNIVALENT FUNCTIONS

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Abstract. Using a notion of neighbourhood of analytic functions due to Stephan Ruscheweyh we examine conditions under which neighbourhoods of a certain class of convex functions are included in a class of starlike functions.

Introduction. Let $A$ denote the class of analytic functions $f$ in the unit disk $E$: \[ \{ z \mid |z| < 1 \} \] with $f(0) = 0$, $f'(0) = 1$. For $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A$ and $\delta > 0$ Ruscheweyh has defined the neighbourhood $N_\delta(f)$ as follows:

\[ N_\delta(f) := \left\{ g(z) = z + \sum_{k=2}^{\infty} b_k z^k \mid \sum_{k=2}^{\infty} k |a_k - b_k| \leq \delta \right\}. \]

He has shown in [1] among other results that if $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in C$ the following result is true:

\[ N_\delta(f) \subseteq S^*, \quad \delta_n = 2^{-2/n}, \]

where $C(S^*)$ denotes the class of normalized convex (starlike) univalent functions in $A$. He also asked if a similar result would hold if we replace $S^*$ by the class

\[ T^* := \left\{ g \in S^* \mid \frac{g'(z)}{g(z)} < 1, z \in E \right\}. \]

and $C$ by the class

\[ T := \left\{ g \in C \mid \frac{g''(z)}{g'(z)} < 1, z \in E \right\}. \]

We prove

Theorem 1. Let $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in A$. Then $N_\delta(f) \subseteq T$, $\delta_n = e^{-1/n}$.

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In [1] Ruscheweyh has shown that for no $\alpha \in [0, 1)$ is there a positive $\delta$ such that
$N_\delta(S^*_\alpha) \subset S^*$. For the class $T$, the situation is quite different as shown by

**Theorem 2.** Let $g(z) := z + \sum_{k=n+1}^{\infty} a_k z^k \in T_r$, $0 \leq r < 1$. Then $N_\delta(g) \subset T$,
$\delta_n = e^{-r/2}(1 - r)$.

The boundaries of $\{w \in \mathbb{C} | \text{Re}[w] > 0\}$ and of $\{w \in \mathbb{C} | \text{Re}[w] > \alpha\}$ are not
 disjoint whereas those of $\{w \in \mathbb{C} | |w - 1| < 1\}$ and of $\{w \in \mathbb{C} | |w - 1| < r\}$ are;
this is one of the reasons for the difference between the two situations. Nevertheless
Theorem 2 is still interesting since the value for $\delta_n$ is best possible.

Concerning this question of boundaries we can prove

**Theorem 3.** Let $f \in T$ and $D := \{zf'(z)/f(z) | z \in E\}$ be such that there is $w \in \overline{D}$
with $|w - 1| = 1$. Then for no $\delta > 0$ we have $N_\delta(f) \subset T$.

It should be noted that no similar result holds if the class $T$ is replaced by the class
$S_*^*$; in fact for $f(z) = z/(1 - z) \in C \subset S^*$ we have $N_{1/4}(f) \subset S^*$ even though the
region $D = \{w \in \mathbb{C} | \text{Re}[w] > 1/2\}$ is such that the point at infinity belongs to both $\overline{D}$
and $\{w \in \mathbb{C} | |w| = 0\}$.

**Proof of Theorem 1.** It was established in [2] that for $f(z) := z + \sum_{k=2}^{\infty} a_k z^k \in T$
we have the estimate $|z| e^{-|z|/n} \leq |f(z)| \leq |z| e^{\sqrt{2}|z|/n}$; using the same method it is very easy
to show that for $f(z) := z + \sum_{k=n+1}^{\infty} a_k z^k \in T$ the estimate

$$|z| e^{-|z|/n} \leq |f(z)| \leq |z| e^{\sqrt{2}|z|/n}$$

is true and sharp as seen from the function $f(z) := z e^{\sqrt{2}/n}$. We also remark that
$f(z) \in \bar{T} \iff sf'(z) \in T$ so that we obtain for $f(z) := z + \sum_{k=n+1}^{\infty} a_k z^k \in \bar{T}$ for the
following estimate

$$e^{-|z|/n} \leq |f'(z)| \leq e^{\sqrt{2}|z|/n}$$

and the sharpness is established by looking at the function $f(z) := \int_0^z e^{u}/u \, du$.

We also remark the following: a function $g(z) \in A$ belongs to the class $T$ iff for
every $\theta \in [0, 2\pi)$ we have

$$z \frac{g'(z)}{g(z)} - 1 \neq e^{i\theta}, \quad z \in E,$$

that is

$$\frac{1}{z} \left( \frac{z/(1 - z)^2 - (1 + e^{i\theta})z/(1 - z)}{-e^{i\theta}} \right) \ast g(z) \neq 0, \quad \theta \in [0, 2\pi), z \in E,$$

where $\ast$ denotes the Hadamard product of two functions. Since

$$-e^{i\theta} h_\theta(z) := \frac{z}{(1 - z)^2} - (1 + e^{i\theta}) \frac{z}{1 - z} = -e^{i\theta} z + \sum_{n=2}^{\infty} (n - 1 - e^{i\theta}) z^n$$
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where \( |n - 1 - e^{i\theta}| \leq n \) it is clear from the results in [1] that a sufficient condition in order that \( N_{\delta_n}(f) \subset T \) may hold for some function \( f \) in \( A \) is that

\[
|\frac{h_\theta(z) * f(z)}{z}| \geq \delta_n, \quad z \in E, \theta \in [0, 2\pi).
\]

Now let \( f(z) := z + \sum_{k=n+1}^\infty a_k z^k \in \tilde{T} \). We have

\[
f(z) * h_\theta(z) = \frac{-e^{i\theta}}{1 - e^{i\theta}} f'(z) - (1 + e^{i\theta}) f(z),
\]

\[
\frac{(f(z) * h_\theta(z))'}{f'(z)} = 1 - e^{-i\theta} \frac{f''(z)}{f'(z)}
\]

with \( \text{Re}[1 - e^{i\theta} z f''(z)/f'(z)] \geq 1 - |zf''(z)/f'(z)| > 0 \). This shows, since \( f \in \tilde{T} \subset C \), that the functions \( h_\theta(z) * f(z) \) are close-to-convex univalent. We also get the estimate

\[
|f(z) * h_\theta(z)| \geq \int_0^{|z|} e^{-u^{n}/n} |(1 - u^n)| \, du = |z| e^{-|z|^n/n}
\]

so that according to (3), \( N_{\delta_n}(f) \subset T \) for \( \delta_n = e^{-1/n} \). The sharpness of the result is seen from the function \( f(z) := \int_0^z e^{u^n/n} \, du \); in fact \( g(z) := f(z) + \delta_n z^{n+1}/(n+1) \in N_{\delta_n}(f) \) and \( g'(z) = f'(z) + \delta_n z^n = 0 \) if \( z^n = -1 \). This completes the proof of Theorem 1.

Proof of Theorem 2. The proof of Theorem 2 is more direct. We first remark that from the definition of \( T \), we have

\[
g(z) \in T \iff g(z) = z \left( \frac{g_1(z)}{z} \right)' \quad \text{for some function } g_1 \in T
\]

so that if \( g(z) := z + \sum_{k=n+1}^\infty a_k z^k \in T \), we get from (1) and Schwarz lemma that

\[
e^{-r|z|^n/n} \leq \left| \frac{g(z)}{z} \right| \leq e^{r|z|^n/n},
\]

\[
|z \frac{g'(z)}{g(z)}| \leq r |z|^n.
\]

Now let \( 0 < \theta < 2\pi \); we have, according to (4) and (5), for \( z \in E \),

\[
|\frac{g(z) * h_\theta(z)}{z}| = |g'(z) - (1 + e^{i\theta}) \frac{g(z)}{z}| \geq \left| \frac{g(z)}{z} \left( 1 - \left| \frac{g'(z)}{g(z)} \right| \right) \right|
\]

\[
\geq e^{-r|z|^n/n} (1 - r |z|^n)
\]

from which it follows, according to (3), that \( N_{\delta_n}(g) \subset T \) for \( \delta_n = (1 - r)e^{-r/n} \). The sharpness of the result is seen from the function \( g(z) := ze^{r|z|^n/n} \); in fact,
\( f(z) := g(z) + \delta_n z^{n+1}/(n + 1) \in N_\delta(g) \) and \( f'(z) = 0 \) if \( z^n = -1 \). This completes the proof of Theorem 2.

**Proof of Theorem 3.** Let \( h_\theta(z) \) be defined as before. Since

\[
\frac{f(z) \ast h_\theta(z)}{z} = -e^{i\theta} \frac{f(z)}{z} \left( z \frac{f'(z)}{f(z)} - 1 \right) e^{i\theta}
\]

it is clear from the hypothesis on \( D \), that, \( |f(z)/z| \) being bounded in \( E \),

\[
\inf \left| \frac{f(z) \ast h_\theta(z)}{z} \right| = 0
\]

where the inf is taken over all \( z \in E, \theta \in [0, 2\pi) \).

We now proceed to show Theorem 3 following an idea due to Ruscheweyh [1]. Let \( \delta > 0 \) and \( n \) some integer > 2. Choose a point \( z_0 \in E \) and \( \theta \in [0, 2\pi) \) such that for \( \mu := (f \ast h_\theta(z_0))/z_0^n \) we have

\[
|\mu| = \left| \frac{f \ast h_\theta(z_0)}{z_0^n} \right| < \delta \left( \frac{n - 2}{n} \right).
\]

This is always possible because of (6) and the fact that the function \( f(z) \ast h_\theta(z) \), \( f \) being in the class \( T \), is nonvanishing for \( z \neq 0 \). We then define the function \( g(z) := f(z) - \mu z^n/a_n \) where \( a_n := h_\theta^{(n)}(0)/n! = (n - 1 - e^{i\theta})/(-e^{i\theta}) \), it is clear that \( |a_n| \geq n - 2 \) so that \( n |\mu/a_n| \leq n |\mu|/(n - 2) < \delta \) and \( g \in N_\delta(f) \); but on the other side we have

\[
\frac{g \ast h_\theta(z_0)}{z_0} = \frac{f \ast h_\theta(z_0)}{z_0} - \mu z_0^{n-1} = 0
\]

which shows that \( g \notin T \). This completes the proof of Theorem 3.

**References**


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