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SMALL TRANSITIVE LATTICES

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ABSTRACT. Partial results are obtained on the problem of determining the smallest lattice of subspaces of a Hilbert space with the property that the only operators leaving all the subspaces invariant are the multiples of the identity.

Halmos [3] initiated the study of *transitive* lattices of subspaces of Hilbert space, i.e., subspace lattices with the property that the only (bounded) operators leaving all the subspaces in the lattice invariant are scalar multiples of the identity operator. By definition, every subspace lattice contains the two trivial elements, $\{0\}$ and the entire space. Halmos [3] gave an example of a transitive lattice of subspaces with only 5 nontrivial elements, and in [4] (cf. [7, §4.7]) an example was constructed with only 4 nontrivial elements.

It is easily seen that there is no transitive subspace lattice with only two nontrivial elements. The question remains: is there a transitive lattice of subspaces with only three nontrivial elements? We have not been able to answer this question, but we show (Theorem 2) that an affirmative answer would follow from the existence of a pair of operator ranges that are simultaneously left invariant by no nonscalar operator. In addition, we construct (Corollary 3) a pair of linear manifolds that are simultaneously left invariant by no nonscalar operator.

In what follows, a collection of linear manifolds in a complex Banach space is *transitive* if the only operators (i.e., bounded linear transformations) leaving all of the manifolds of the collection invariant are the scalars. We use the term *linear transformation* to refer to a possibly unbounded transformation defined on the entire space. The word *dimension* always refers to algebraic (linear, Hamel) dimension. The symbol c denotes the cardinality of the continuum. Our methods are heuristically inspired by those of Shields [8].

LEMMA. *Suppose that X is a separable infinite-dimensional Banach space, M is a linear manifold in X, and A: $X \to X$ is a linear transformation with the property that, for every x in X, the vector Ax is a multiple of x modulo M. Then there is a linear transformation F: $X \to M$ and a scalar λ such that $A = \lambda + F$. Also, if A is bounded and M has dimension less than c, then F has finite rank.*

PROOF. If is clear from the hypothesis that A leaves M invariant. Since every nonzero vector is an eigenvector for the quotient operator on X/M induced by A,

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this quotient operator is a scalar multiple, say λ, of the identity on X/M. Clearly the range of $A-\lambda$ is contained in M. Let $F=A-\lambda$. If A is bounded, then F is bounded, so the dimension of the range of F is either finite or c. If the dimension of M is less than c, F must have finite rank.

THEOREM 1. *If X is a separable infinite-dimensional Banach space, then there is a linear transformation $T\colon X \to X$ satisfying $T^2=1$ such that whenever A and B are bounded operators and $AT=TB$ there is a scalar λ such that $A=B=\lambda$.*

PROOF. Let β denote the smallest ordinal number with c predecessors. Consider the collection of all ordered pairs (C, D) of bounded operators such that neither is a scalar and D is not invertible modulo the compact operators. The cardinality of this set is c, so we can write this set as $\{(C_\alpha, D_\alpha)\colon \alpha<\beta\}$. Below, we use transfinite recursion to construct a family $\{T_\alpha\colon \alpha<\beta\}$ of linear transformations, a family $\{M_\alpha\colon \alpha<\beta\}$ of linear submanifolds of X, and a family $\{x_\alpha\colon \alpha<\beta\}$ of vectors such that, for every $\alpha<\beta$:

(1) $T_\alpha\colon M_\alpha \to M_\alpha$ and $T_\alpha^2=1$,

(2) $\dim M_\alpha \leqslant \aleph_0 + \operatorname{card}\alpha$,

(3) if $\alpha<\mu<\beta$, then $M_\alpha \subset M_\mu$ and $T_\mu \mid M_\alpha = T_\alpha$,

(4) $x_\alpha \in M_\alpha$, $D_\alpha x_\alpha \in M_\alpha$, and $C_\alpha T_\alpha x_\alpha \neq T_\alpha D_\alpha x_\alpha$.

Before constructing the above families, we indicate how the theorem follows from the construction. Suppose then, that the above families are given. Define T on X as follows. On $\bigcup_{\alpha<\beta} M_\alpha$ define T by $T \mid M_\alpha = T_\alpha$. Choose any algebraic complement L of $\bigcup_{\alpha<\beta} M_\alpha$ and define T to be the identity on L. Then T is a linear transformation on X and it follows from (1) that $T^2=1$. Suppose that A and B are bounded operators and $AT=TB$. Since T is invertible, it follows that $A=B$ if either A or B is a scalar. If neither A nor B is a scalar, then, since $(A+\lambda)T=T(B+\lambda)$, we can assume that B is not invertible modulo the compact operators. Then $(A, B)=(C_\alpha, D_\alpha)$ for some $\alpha<\beta$, and this contradicts (4).

To prove the theorem, then, we must construct families satisfying (1)–(4) above. Let γ be any ordinal less than β, and suppose that the M_α's, T_α's, and x_α's have been constructed for all $\alpha<\gamma$ so that (1)–(4) hold when β is replaced by γ. We construct M_γ, T_γ, and x_γ as follows.

Let $M=\bigcup_{\alpha<\gamma} M_\alpha$ and let $S\colon M \to M$ be defined by $S \mid M_\alpha = T_\alpha$ for all $\alpha<\gamma$. We consider two cases; the lemma implies that these cases are exhaustive.

Case 1. There is an x in X such that $D_\gamma x$ is not a multiple of x modulo M. In this case let M_γ be the manifold spanned by M, x, and $D_\gamma x$; we need to define $T_\gamma\colon M_\gamma \to M_\gamma$. Let $T_\gamma \mid M = S$. If $x \in M$, then $T_\gamma x$ is defined; if $x \notin M$, then we let $T_\gamma x = x$. Now we define $T_\gamma D_\gamma x$ to be either $D_\gamma x$ or $-D_\gamma x$ so that $T_\gamma D_\gamma x \neq C_\gamma T_\gamma x$. Finally, we let $x_\gamma = x$.

Case 2. There is a scalar λ such that $D_\gamma - \lambda$ has finite rank and range contained in M. Since D_γ is not invertible modulo the compact operators, λ must be 0. Choose any x not in M such that $D_\gamma x \neq 0$. Similarly, choose a y not in the span of M and x so that $C_\gamma y \neq 0$. Also, by replacing y by $-y$ if necessary, we can assume that $C_\gamma y \neq SD_\gamma x$. Then let $x_\gamma = x$, let M_γ be the linear manifold spanned by M, x, and y, and define $T_\gamma\colon M_\gamma \to M_\gamma$ as the extension of S satisfying $T_\gamma x = y$ and $T_\gamma y = x$.

The above construction can also be used to produce the initial M_0, T_0, and x_0. For this let $M = \{0\}$; since D_0 is not a scalar, Case 1 applies.

By the principle of transfinite recursion, we have constructed M_α, T_α, and x_α for all $\alpha < \beta$ so that (1)–(4) hold.

COROLLARY 1. *There is a transitive lattice of linear manifolds in a separable infinite-dimensional Hilbert space (and in any separable infinite-dimensional Banach space of the form $X \oplus X$) that has three nontrivial elements, two of which are closed.*

PROOF. Let H be a separable infinite-dimensional Hilbert space. The lattice consists of $\{0\}$, $H \oplus H$, $\{0\} \oplus H$, $H \oplus \{0\}$, and $\{x \oplus Tx: x \in H\}$, where T is a linear transformation as in Theorem 1. If S is a bounded linear operator on $H \oplus H$ leaving all of these manifolds invariant, then S has the form $B \oplus A$ with B and A bounded operators on H. The invariance of $\{x \oplus Tx: x \in H\}$ implies $AT = TB$, and it follows from the choice of T (Theorem 1) that S is a scalar.

COROLLARY 2. *If X is a separable infinite-dimensional Banach space, then there is a linear transformation $P: X \to X$ such that $P^2 = P$, and the only bounded operators A for which $AP = PA$ are the scalars.*

PROOF. Let T be a transformation as in Theorem 1, and define $P = (1 + T)/2$. Then $T^2 = 1$ implies $P^2 = P$. If A is an operator and $AP = PA$, then $AT = TA$, so A is a scalar by Theorem 1.

COROLLARY 3. *If X is a separable infinite-dimensional Banach space, then there is a transitive lattice of linear manifolds that has two nontrivial elements.*

PROOF. Let P be as in Corollary 2; then the lattice with elements X, $\{0\}$, kernel(P), and range(P) is transitive.

A linear manifold is called an *operator range* if there is a bounded linear operator on the whole space whose range is the given manifold. Most of the known results concerning operator ranges are discussed in the elegant survey paper [**1**]. Foias [**2**] initiated the study of invariant operator ranges; recent work in this area includes [**5**] and [**6**].

The following theorem shows that if Corollary 3 could be strengthened to the existence of a transitive set with two nontrivial elements, both of which were dense operator ranges, then there would be a transitive subspace lattice with three nontrivial elements.

THEOREM 2. *If there exist two dense operator ranges in Hilbert space such that the only bounded operators leaving both of them invariant are the scalars, then there is a transitive subspace lattice with three nontrivial elements.*

PROOF. As observed in [**1**], we can assume that the operator ranges are the ranges of positive operators P_1 and P_2 on the Hilbert space H. Let L denote the following collection of subspaces of $H \oplus H$: $\{0\}$, $H \oplus H$, $\{x \oplus P_1 x: x \in H\}$, $\{x \oplus P_2 x: x \in H\}$, and $H \oplus \{0\}$. We claim that L is a transitive lattice. Note that the ranges of P_1 and P_2 intersect only in $\{0\}$, for otherwise any nonscalar operator with range contained in the intersection would leave both ranges invariant. Thus $P_1 - P_2$ is injective and hence has dense range; it easily follows that L is a subspace lattice. Suppose that T leaves the elements of L invariant. Write T as an operator matrix on $H \oplus H$:

$$T = \begin{pmatrix} A & B \\ Y & C \end{pmatrix}.$$

Since $H \oplus \{0\} \in L$, $Y = 0$. Then

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \begin{pmatrix} x \\ P_i x \end{pmatrix} = \begin{pmatrix} (A + BP_i)x \\ CP_i x \end{pmatrix} = \begin{pmatrix} (A + BP_i)x \\ P_i(A + BP_i)x \end{pmatrix}$$

yields $CP_i = P_i(A + BP_i)$ for $i = 1, 2$. Thus C leaves the ranges of both P_i's invariant, and hence $C = \lambda$ for some scalar λ. By replacing T with $T - \lambda$, we can assume that $C = \lambda = 0$. Then $P_i(A + BP_i) = 0$, and since each P_i is injective, we conclude that $A + BP_1 = A + BP_2 = 0$. Hence $B(P_1 - P_2) = 0$, so $B = 0$ and then $A = 0$.

REMARKS. 1. There exist two linear manifolds that form a transitive set and there may exist two operator ranges that do. However, neither element of a two-element transitive set can be closed or even nondense. If M is any nondense manifold and N is a nonzero manifold, then the Hahn-Banach theorem insures the existence of a bounded operator with rank 1 that annihilates M and whose range is included in N. Such an operator leaves both M and N invariant.

2. It is clear that a single linear manifold cannot form a transitive set. However, by imitating the proof of Theorem 1 it is possible to construct a linear manifold in each separable infinite-dimensional Banach space such that the only bounded operators that leave the manifold invariant are of the form $\lambda + F$ where λ is a scalar and F is a finite rank operator whose range is included in the manifold.

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