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THE BAIRE CLASS OF APPROXIMATE SYMMETRIC DERIVATES

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ABSTRACT. It is shown that all approximate symmetric derivatives of measurable functions are in Baire class one. Further, if f is a measurable function which is finite a.e., then its upper and lower approximate symmetric derivates are in Baire class three.

An extended real-valued function, f, defined on $\mathbf{R}$ is said to be in the first Baire class, abbreviated $\mathcal{B}_1$, if there is a sequence of continuous functions, $f_1, f_2, \ldots$, such that $f_n(x) \to f(x)$ for all x. In a similar way, f is in the second Baire class, abbreviated $\mathcal{B}_2$, if there is a sequence of functions, $f_1, f_2, \ldots$, each in $\mathcal{B}_1$, such that $f_n(x) \to f(x)$ for all x. It is clear that this process can be continued to generate a class of functions, $\mathcal{B}_\alpha$, corresponding to each ordinal number α (see Goffman [**1**]). Our purpose here is to show that the upper and lower approximate symmetric derivates of any measurable function which is finite a.e. are in $\mathcal{B}_3$ and if it is symmetrically differentiable, then its approximate symmetric derivative is in $\mathcal{B}_1$.

Let $A \subset \mathbf{R}$. The reflection of A through some $x \in \mathbf{R}$ will be denoted $R_x(A)$. If A is measurable, then its measure is denoted by $|A|$. The upper density of A at $x \in \mathbf{R}$ is written

$$\bar{d}(A, x) = \limsup_{h,k \to 0+} \frac{|A \cap (x-h, x+k)|}{h+k};$$

the lower density of A at x, $\underline{d}(A, x)$, and the density of A at x, $d(A, x)$, are written similarly.

Let f be a measurable function defined on $\mathbf{R}$. If $x, t \in \mathbf{R}$ such that $t \neq 0$ and $f(x+t) - f(x-t)$ is defined, then we write

$$q(x, t) = \frac{f(x+t) - f(x-t)}{2t}.$$

The upper approximate symmetric derivate of f at x is

$$\bar{f}^s_{\mathrm{ap}}(x) = \inf\{a: \bar{d}(\{t: q(x,t) > a\}, 0) = 0\};$$

the lower approximate symmetric derivate of f at x, $\underline{f}^s_{\mathrm{ap}}(x)$, is defined analogously. If $\bar{f}^s_{\mathrm{ap}}(x) = \underline{f}^s_{\mathrm{ap}}(x)$, whether finite or infinite, then f is approximately symmetrically differentiable at x and the common value of the derivates is written $f^s_{\mathrm{ap}}(x)$.

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LEMMA. *Let f be a finite-valued measurable function defined on* $\mathbf{R}$. *If* $\alpha > 0$, $\beta \in \mathbf{R}$ *and* $\gamma \in (0, 1)$, *then the sets*

$$A_f(\alpha, \beta, \gamma) = \{x: |\{t \in (0, \alpha): q(x, t) > \beta\}| > \alpha\gamma\}$$

and

$$B_f(\alpha, \beta, \gamma) = \{x: |\{t \in (0, \alpha): q(x, t) < \beta\}| > \alpha\gamma\}$$

are both open.

PROOF. No generality is lost with the assumption that $\beta = 0$, for if $\beta \neq 0$ we may consider $g(x) = f(x) - \beta x$ in which case

$$A_f(\alpha, \beta, \gamma) = A_g(\alpha, 0, \gamma).$$

Suppose $0 \in A_f(\alpha, 0, \gamma)$ and let C be the set of all $t \in (0, \alpha)$ such that:
(i) f is approximately continuous at t and $-t$ and
(ii) $f(t) > f(-t)$.
Then $|C| > \alpha\gamma$ because of the definition of $A_f(\alpha, 0, \gamma)$ and because the measurability of f implies that it is approximately continuous a. e. (see Goffman [**1**, p. 190]). For each positive integer, n, define

$$C_n = \{t \in C: f(t) - f(-t) > \tfrac{1}{n}\}.$$

It is clear that C_n is measurable for each n and that $C = \bigcup_{n=1}^{\infty} C_n$. Therefore there exists an integer, m, large enough so that

$$|C_m| > \alpha\delta > \alpha\gamma$$

for some $\delta > \gamma$. Since f is approximately continuous at t and $-t$ whenever $t \in C_m$, we may choose $r(t) > 0$ such that when $0 < h < r(t)$

$$\left|\left\{x \in [t - h, t + h]: |f(x) - f(t)| < \frac{1}{2m}\right\}\right| > 2h\left(\frac{3\gamma + 5\delta}{8\delta}\right) \tag{1}$$

and

$$\left|\left\{x \in [-t - h, -t + h]: |f(x) - f(t)| < \frac{1}{2m}\right\}\right| > 2h\left(\frac{3\gamma + 5\delta}{8\delta}\right). \tag{2}$$

Define

$$\Lambda = \{[t - h, t + h]: t \in C_m, 0 < h < \min\{r(t), t, \alpha - t\}\}.$$

Λ is a Vitali cover for C_m, so there exists a sequence, $\{I_n\} \subset \Lambda$, such that $I_\mu \cap I_\nu = \varnothing$ whenever $\mu \neq \nu$ and

$$\left|C_m - \bigcup_{n=1}^{\infty} I_n\right| = 0.$$

Thus, we may choose an integer, N, large enough so that

$$\left|\bigcup_{n=1}^{N} I_n\right| > \alpha\delta.$$

Define t_n to be the center of I_n, $J_n = R_0(I_n)$,

$$R_n = \left\{t \in I_n: |f(t) - f(t_n)| < \frac{1}{2m}\right\}$$

and

$$S_n = \left\{ t \in J_n : |f(t) - f(-t_n)| < \frac{1}{2m} \right\}.$$

According to (1) and (2),

$$(3) \qquad |R_n| > |I_n| \frac{3\gamma + 5\delta}{8\delta} \quad \text{and} \quad |S_n| > |J_n| \frac{3\gamma + 5\delta}{8\delta}.$$

Note that if $u \in R_n$ and $v \in S_n$ for some fixed n, then

$$(4) \qquad \begin{aligned} f(u) - f(v) &> \left(f(t_n) - \frac{1}{2m} \right) - \left(f(-t_n) + \frac{1}{2m} \right) \\ &= f(t_n) - f(-t_n) - \frac{1}{m} > 0 \end{aligned}$$

because $t_n \in C_m$.

Choose ε such that

$$0 < \varepsilon < \frac{\delta - \gamma}{8\delta} \min_{1 \leqslant n \leqslant N} |I_n|$$

and let $|x| < \varepsilon$. Then, using (4) and (3) we see that

$$\begin{aligned} |\{t \in (0, \alpha) : f(x + t) > f(x - t)\}| &\geqslant \left| \bigcup_{n=1}^{N} R_x(S_n) \cap R_n \right| \\ &= \sum_{n=1}^{N} |R_x(S_n) \cap R_n| \\ &\geqslant \sum_{n=1}^{N} [|R_x(S_n)| + |R_n| - 2|x| - |I_n|] \\ &> \sum_{n=1}^{N} \left[|J_n| \left(\frac{3\gamma + 5\delta}{8\delta} \right) + |I_n| \left(\frac{3\gamma + 5\delta}{8\delta} \right) - \left(\frac{\delta - \gamma}{4\delta} \right) |I_n| - |I_n| \right] \\ &= \frac{\gamma}{\delta} \sum_{n=1}^{N} |I_n| > \alpha\gamma. \end{aligned}$$

Therefore $x \in A_f(\alpha, 0, \gamma)$ and it follows that $(-\varepsilon, \varepsilon) \subset A_f(\alpha, 0, \gamma)$. If $s \in A_f(\alpha, 0, \gamma)$, then by considering $g(x) = f(x + s)$ it can be shown in a manner similar to the above that there is an $\varepsilon > 0$ such that $(s - \varepsilon, s + \varepsilon) \subset A_f(\alpha, 0, \gamma)$. Therefore $A_f(\alpha, 0, \gamma)$ is open.

The proof that $B_f(\alpha, \beta, \gamma)$ is open follows from the observation that $B_f(\alpha, \beta, \gamma) = A_{-f}(\alpha, -\beta, \gamma)$.

THEOREM 1. *If f is a measurable function such that f^s_{ap} exists everywhere, then $f^s_{\text{ap}} \in \mathfrak{B}_1$.*

PROOF. Let $N^+ = \{x : f(x) = \infty\}$ and $N^- = \{x : f(x) = -\infty\}$. If $|N^+| > 0$, then there is an $x \in \mathbf{R}$ such that $d(N^+, x) = 1$. It follows easily that

$$d(\{t : q(x, t) \text{ is defined}\}, 0) = 0$$

which implies that $\underline{f}^s_{\mathrm{ap}}(x) = -\infty$ and $\bar{f}^s_{\mathrm{ap}}(x) = \infty$. But, $f^s_{\mathrm{ap}}(x)$ exists so this contradiction leads us to conclude that $|N^+| = 0$. It can be shown similarly that $|N^-| = 0$. Therefore f is finite a.e. Since the definition of $f^s_{\mathrm{ap}}(x)$ does not depend upon $f(x)$, no generality is lost with the assumption that f is real-valued everywhere.

Suppose $\beta \in \mathbf{R}$, $\gamma \in (\frac{1}{2}, 1)$, m is a positive integer and

$$x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_f\left(\frac{1}{n}, \beta - \frac{1}{m}, \gamma\right).$$

Then there is a sequence of integers, $n_1, n_2, \ldots$, such that $x \in A_f\left(\frac{1}{n_i}, \beta - \frac{1}{m}, \gamma\right)$ for each i. Thus,

$$\left|\left\{t \in \left(0, \frac{1}{n_i}\right): q(x, t) > \beta - \frac{1}{m}\right\}\right| > \frac{\gamma}{n_i}$$

and it follows at once that

$$\bar{d}\left(\{t: q(x, t) > \beta - \tfrac{1}{m}\}, 0\right) \geqslant \gamma > \tfrac{1}{2}.$$

From the definition of f^s_{ap} we see that $f^s_{\mathrm{ap}}(x) > \beta - \frac{1}{m}$. Therefore

$$\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_f\left(\frac{1}{n}, \beta - \frac{1}{m}, \gamma\right) \subset \left\{x: f^s_{\mathrm{ap}}(x) > \beta - \frac{1}{m}\right\},$$

which implies

$$\bigcap_{m=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_f\left(\frac{1}{n}, \beta - \frac{1}{m}, \gamma\right) \subset \{x: f^s_{\mathrm{ap}}(x) \geqslant \beta\}. \tag{5}$$

Now let $f^s_{\mathrm{ap}}(x) \geqslant \beta$. Then for each positive integer, m, there is a positive integer, N, such that when $n \geqslant N$,

$$\left|\left\{t \in \left(0, \frac{1}{n}\right): q(x, t) > \beta - \frac{1}{m}\right\}\right| > \frac{\gamma}{n}.$$

Therefore

$$x \in \bigcup_{n=k}^{\infty} A_f\left(\frac{1}{n}, \beta - \frac{1}{m}, \gamma\right)$$

for all k. This implies that

$$x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_f\left(\frac{1}{n}, \beta - \frac{1}{m}, \gamma\right).$$

Since $f^s_{\mathrm{ap}}(x) \geqslant \beta$, this same reasoning holds for any positive integer, m, and so

$$x \in \bigcap_{m=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_f\left(\frac{1}{n}, \beta - \frac{1}{m}, \gamma\right). \tag{6}$$

In light of (5) and (6) we conclude that

$$\{x: f^s_{\mathrm{ap}}(x) \geqslant \beta\} = \bigcap_{m=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_f\left(\frac{1}{n}, \beta - \frac{1}{m}, \gamma\right).$$

It follows similarly that

$$\{x: f^s_{\text{ap}}(x) \leqslant \beta\} = \bigcap_{m=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} B_f\left(\frac{1}{n}, \beta + \frac{1}{m}, \gamma\right).$$

Now we apply the lemma to see that both

$$\{x: f^s_{\text{ap}}(x) \geqslant \beta\} \quad \text{and} \quad \{x: f^s_{\text{ap}}(x) \leqslant \beta\}$$

are G_δ sets. Using the standard theorem in Goffman [**1**, p. 141], it follows that $f^s_{\text{ap}} \in \mathcal{B}_1$.

THEOREM 2. *Let f be a measurable function which is real-valued a. e. and let* $\beta \in [-\infty, \infty]$. *Then the sets*

$$\{x: \bar{f}^s_{\text{ap}}(x) > \beta\} \quad \textit{and} \quad \{x: \underline{f}^s_{\text{ap}}(x) < \beta\}$$

are both $G_{\delta\sigma}$ *sets and the sets*

$$\{x: \bar{f}^s_{\text{ap}}(x) < \beta\} \quad \textit{and} \quad \{x: \underline{f}^s_{\text{ap}}(x) > \beta\}$$

are both $F_{\sigma\delta\sigma}$ *sets.*

PROOF. As before we may assume that f is real-valued everywhere. First, assume $\beta \in \mathbf{R}$. Let $\gamma \in (0, 1)$ and

$$x \in \bigcap_{k=1}^{\infty} \bigcup_{0<\alpha<1/k} A_f(\alpha, \beta, \gamma).$$

Then for each positive integer, k, there is an $\alpha_k \in (0, \frac{1}{k})$ such that

$$|\{t \in (0, \alpha_k): q(x, t) > \beta\}| > \alpha_k \gamma.$$

Therefore

$$\bar{d}(\{t: q(x, t) > \beta\}, 0) \geqslant \gamma$$

and we conclude that $\bar{f}^s_{\text{ap}}(x) > \beta$. From this it follows that if

$$x \in \bigcup_{n=2}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{0<\alpha<1/k} A_f\left(\alpha, \beta, \frac{1}{n}\right),$$

then $\bar{f}^s_{\text{ap}}(x) > \beta$. Thus

$$\bigcup_{n=2}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{0<\alpha<1/k} A_f\left(\alpha, \beta, \frac{1}{n}\right) \subset \{x: \bar{f}^s_{\text{ap}}(x) > \beta\}. \tag{7}$$

Now, suppose $\bar{f}^s_{\text{ap}}(x) > \beta$. Then there is a positive integer, n_0, such that

$$\bar{d}(\{t: q(x, t) > \beta\}, 0) > \tfrac{1}{n_0}.$$

This implies that for each positive integer, k, there is an $\alpha_k \in (0, \frac{1}{k})$ such that $x \in A_f(\alpha_k, \beta, \frac{1}{n_0})$. Therefore

$$x \in \bigcap_{k=1}^{\infty} \bigcup_{0<\alpha<1/k} A_f\left(\alpha, \beta, \frac{1}{n_0}\right) \subset \bigcup_{n=2}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{0<\alpha<1/k} A_f\left(\alpha, \beta, \frac{1}{n}\right)$$

and so

$$(8) \qquad \left\{x : \bar{f}^s_{\mathrm{ap}}(x) > \beta\right\} \subset \bigcup_{n=2}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{0<\alpha<1/k} A_f\left(\alpha, \beta, \frac{1}{n}\right).$$

From (7) and (8) it follows that

$$\left\{x : \bar{f}^s_{\mathrm{ap}}(x) > \beta\right\} = \bigcup_{n=2}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{0<\alpha<1/k} A_f\left(\alpha, \beta, \frac{1}{n}\right).$$

It follows from the lemma that $\{x : \bar{f}^s_{\mathrm{ap}}(x) > \beta\}$ is a $G_{\delta\sigma}$ set.

If $|\beta| = \infty$, then

$$\left\{x : \bar{f}^s_{\mathrm{ap}}(x) > \infty\right\} = \varnothing \quad \text{and} \quad \left\{x : \bar{f}^s_{\mathrm{ap}}(x) > -\infty\right\} = \bigcup_{n=1}^{\infty} \left\{x : \bar{f}^s_{\mathrm{ap}}(x) > -n\right\}$$

are both $G_{\delta\sigma}$ sets. Note that if $\beta \in \mathbf{R}$, then

$$\left\{x : \bar{f}^s_{\mathrm{ap}}(x) \geqslant \beta\right\} = \bigcap_{n=1}^{\infty} \left\{x : \bar{f}^s_{\mathrm{ap}}(x) > \beta - \frac{1}{n}\right\},$$

$$\left\{x : \bar{f}^s_{\mathrm{ap}}(x) \geqslant \infty\right\} = \bigcap_{n=1}^{\infty} \left\{x : \bar{f}^s_{\mathrm{ap}}(x) > n\right\}$$

and

$$\left\{x : \bar{f}^s_{\mathrm{ap}}(x) \geqslant -\infty\right\} = \mathbf{R},$$

where all three sets are $G_{\delta\sigma\delta}$ sets. Therefore, for any $\beta \in [-\infty, \infty]$,

$$\left\{x : \bar{f}^s_{\mathrm{ap}}(x) < \beta\right\} = \mathbf{R} - \left\{x : \bar{f}^s_{\mathrm{ap}}(x) \geqslant \beta\right\}$$

is an $F_{\sigma\delta\sigma}$ set.

To prove the theorem for the lower derivate, it suffices to note that if $g = -f$, then $\underline{f}^s_{\mathrm{ap}} = -(\bar{g}^s_{\mathrm{ap}})$.

According to Goffman [**1**, p. 141], Theorem 2 immediately implies the following corollary.

COROLLARY 3. *Let f be a measurable function which is finite a.e. Then $\bar{f}^s_{\mathrm{ap}}$ and $\underline{f}^s_{\mathrm{ap}}$ are both in $\mathfrak{B}_3$.*

Corollary 3 is an improvement on a theorem of Kundu [**2**], who showed that if f is continuous, then $\bar{f}^s_{\mathrm{ap}}$ is measurable. It was shown by Preiss [**4**] that all ordinary approximate derivatives are in $\mathfrak{B}_1$ and Larson [**3**] showed that all ordinary symmetric derivatives are in $\mathfrak{B}_1$; Theorem 1 is a partial extension of both of these results.

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