THE BAIRE CLASS OF APPROXIMATE SYMMETRIC DERIVATES

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Abstract. It is shown that all approximate symmetric derivatives of measurable functions are in Baire class one. Further, if $f$ is a measurable function which is finite a.e., then its upper and lower approximate symmetric derivates are in Baire class three.

An extended real-valued function, $f$, defined on $\mathbb{R}$ is said to be in the first Baire class, abbreviated $\mathcal{B}_1$, if there is a sequence of continuous functions, $f_1, f_2, \ldots$, such that $f_n(x) \to f(x)$ for all $x$. In a similar way, $f$ is in the second Baire class, abbreviated $\mathcal{B}_2$, if there is a sequence of functions, $f_1, f_2, \ldots$, each in $\mathcal{B}_1$, such that $f_n(x) \to f(x)$ for all $x$. It is clear that this process can be continued to generate a class of functions, $\mathcal{B}_a$, corresponding to each ordinal number $a$ (see Goffman [1]). Our purpose here is to show that the upper and lower approximate symmetric derivates of any measurable function which is finite a.e. are in $\mathcal{B}_3$ and if it is symmetrically differentiable, then its approximate symmetric derivative is in $\mathcal{B}_1$.

Let $\mathcal{A} \subseteq \mathbb{R}$. The reflection of $\mathcal{A}$ through some $x \in \mathbb{R}$ will be denoted $R_x(\mathcal{A})$. If $\mathcal{A}$ is measurable, then its measure is denoted by $|\mathcal{A}|$. The upper density of $\mathcal{A}$ at $x \in \mathbb{R}$ is written

$$d^+(\mathcal{A}, x) = \limsup_{h,k \to 0^+} \frac{|\mathcal{A} \cap (x-h, x+k)|}{h+k};$$

the lower density of $\mathcal{A}$ at $x$, $d(\mathcal{A}, x)$, and the density of $\mathcal{A}$ at $x$, $d(\mathcal{A}, x)$, are written similarly.

Let $f$ be a measurable function defined on $\mathbb{R}$. If $x, t \in \mathbb{R}$ such that $t \neq 0$ and $f(x + t) - f(x - t)$ is defined, then we write

$$q(x, t) = \frac{f(x + t) - f(x - t)}{2t}.$$  

The upper approximate symmetric derivate of $f$ at $x$ is

$$f_{ap}^+(x) = \inf\{a: d^+(\{t: q(x, t) > a\}, 0) = 0\};$$

the lower approximate symmetric derivate of $f$ at $x$, $f_{ap}^-(x)$, is defined analogously. If $f_{ap}^+(x) = f_{ap}^-(x)$, whether finite or infinite, then $f$ is approximately symmetrically differentiable at $x$ and the common value of the derivates is written $f_{ap}^s(x)$.
LEMMA. Let $f$ be a finite-valued measurable function defined on $\mathbb{R}$. If $\alpha > 0$, $\beta \in \mathbb{R}$ and $\gamma \in (0, 1)$, then the sets

$$A_f(\alpha, \beta, \gamma) = \{x : \{t \in (0, \alpha) : q(x, t) > \beta\} \supset \alpha \gamma\}$$

and

$$B_f(\alpha, \beta, \gamma) = \{x : \{t \in (0, \alpha) : q(x, t) < \beta\} \supset \alpha \gamma\}$$

are both open.

PROOF. No generality is lost with the assumption that $\beta = 0$, for if $\beta \neq 0$ we may consider $g(x) = f(x) - \beta x$ in which case

$$A_f(\alpha, \beta, \gamma) = A_g(\alpha, 0, \gamma).$$

Suppose $0 \in A_f(\alpha, 0, \gamma)$ and let $C$ be the set of all $t \in (0, \alpha)$ such that:

(i) $f$ is approximately continuous at $t$ and $-t$ and

(ii) $f(t) > f(-t)$.

Then $|C| \supset \alpha \gamma$ because of the definition of $A_f(\alpha, 0, \gamma)$ and because the measurability of $f$ implies that it is approximately continuous a.e. (see Goffman [1, p. 190]). For each positive integer, $n$, define

$$C_n = \{t \in C : f(t) - f(-t) > \frac{1}{n}\}.$$

It is clear that $C_n$ is measurable for each $n$ and that $C = \bigcup_{n=1}^{\infty} C_n$. Therefore there exists an integer, $m$, large enough so that

$$|C_m| \supset \alpha \delta > \alpha \gamma$$

for some $\delta > \gamma$. Since $f$ is approximately continuous at $t$ and $-t$ whenever $t \in C_m$, we may choose $r(t) > 0$ such that when $0 < h < r(t)$

$$\left|x \in [t - h, t + h] : |f(x) - f(t)| \leq \frac{1}{2m}\right| \supset 2h \left(\frac{3\gamma + 5\delta}{8\delta}\right)$$

and

$$\left|x \in [-t - h, -t + h] : |f(x) - f(t)| \leq \frac{1}{2m}\right| \supset 2h \left(\frac{3\gamma + 5\delta}{8\delta}\right).$$

Define

$$\Lambda = \{[t - h, t + h] : t \in C_m, 0 < h < \min\{r(t), t, \alpha - t\}\}.$$ 

$\Lambda$ is a Vitali cover for $C_m$, so there exists a sequence, $\{I_n\} \subset \Lambda$, such that $I_{\mu} \cap I_{\nu} = \emptyset$ whenever $\mu \neq \nu$ and

$$\left|C_m - \bigcup_{n=1}^{\infty} I_n\right| = 0.$$ 

Thus, we may choose an integer, $N$, large enough so that

$$\left|\bigcup_{n=1}^{N} I_n\right| \supset \alpha \delta.$$

Define $t_n$ to be the center of $I_n$, $J_n = R(t_n)$.

$$R_n = \{t \in I_n : |f(t) - f(t_n)| \leq \frac{1}{2m}\}$$
and

\[ S_n = \left\{ t \in J_n : \left| f(t) - f(-t_n) \right| < \frac{1}{2m} \right\}. \]

According to (1) and (2),

\[ |R_n| > |I_n| \frac{3\gamma + 5\delta}{8\delta} \quad \text{and} \quad |S_n| > |J_n| \frac{3\gamma + 5\delta}{8\delta}. \]

Note that if \( u \in R_n \) and \( v \in S_n \) for some fixed \( n \), then

\[ f(u) - f(v) > \left( f(t_n) - \frac{1}{2m} \right) - \left( f(-t_n) + \frac{1}{2m} \right) = f(t_n) - f(-t_n) - \frac{1}{m} > 0 \]

because \( t_n \in C_m \).

Choose \( \varepsilon \) such that

\[ 0 < \varepsilon < \frac{\delta - \gamma}{8\delta} \min_{1 \leq n \leq N} |I_n| \]

and let \( |x| < \varepsilon \). Then, using (4) and (3) we see that

\[ \left| \{ t \in (0, \alpha) : f(x + t) > f(x - t) \} \right| \geq \left| \bigcup_{n=1}^{N} R_x(S_n) \cap R_n \right| \]

\[ = \sum_{n=1}^{N} |R_x(S_n) \cap R_n| \]

\[ \geq \sum_{n=1}^{N} \left[ |R_x(S_n)| + |R_n| - 2|x| - |I_n| \right] \]

\[ > \sum_{n=1}^{N} \left[ |J_n| \left( \frac{3\gamma + 5\delta}{8\delta} \right) + |I_n| \left( \frac{3\gamma + 5\delta}{8\delta} \right) - \left( \frac{\delta - \gamma}{4\delta} \right) |I_n| - |I_n| \right] \]

\[ = \frac{\gamma}{\delta} \sum_{n=1}^{N} |I_n| > \alpha \gamma. \]

Therefore \( x \in A_f(\alpha, 0, \gamma) \) and it follows that \( (-\varepsilon, \varepsilon) \subset A_f(\alpha, 0, \gamma) \). If \( s \in A_f(\alpha, 0, \gamma) \), then by considering \( g(x) = f(x + s) \) it can be shown in a manner similar to the above that there is an \( \varepsilon > 0 \) such that \( (s - \varepsilon, s + \varepsilon) \subset A_f(\alpha, 0, \gamma) \). Therefore \( A_f(\alpha, 0, \gamma) \) is open.

The proof that \( B_f(\alpha, \beta, \gamma) \) is open follows from the observation that \( B_f(\alpha, \beta, \gamma) = A_{-f}(\alpha, -\beta, \gamma) \).

**THEOREM 1.** If \( f \) is a measurable function such that \( f_{ap}^{+} \) exists everywhere, then \( f_{ap}^{+} \in \mathcal{B}_{1} \).

**Proof.** Let \( N^{+} = \{ x : f(x) = \infty \} \) and \( N^{-} = \{ x : f(x) = -\infty \} \). If \( |N^{+}| > 0 \), then there is an \( x \in \mathbb{R} \) such that \( d(N^{+}, x) = 1 \). It follows easily that

\[ d(\{ t : q(x, t) \text{ is defined} \}, 0) = 0 \]
which implies that \( f_{ap}^* (x) = -\infty \) and \( f_{ap}^* (x) = \infty \). But, \( f_{ap}^* (x) \) exists so this contradiction leads us to conclude that \( |N^+| = 0 \). It can be shown similarly that \( |N^-| = 0 \). Therefore \( f \) is finite a.e. Since the definition of \( f_{ap}^* (x) \) does not depend upon \( f(x) \), no generality is lost with the assumption that \( f \) is real-valued everywhere.

Suppose \( \beta \in \mathbb{R}, \gamma \in (\frac{1}{2}, 1) \), \( m \) is a positive integer and
\[
x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_f \left( \frac{1}{n}, \beta - \frac{1}{m}, \gamma \right).
\]
Then there is a sequence of integers, \( n_1, n_2, \ldots \), such that \( x \in A_f \left( \frac{1}{n_i}, \beta - \frac{1}{m}, \gamma \right) \) for each \( i \). Thus,
\[
\left| \left\{ t \in \left( 0, \frac{1}{n_i} \right) : q(x, t) > \beta - \frac{1}{m} \right\} \right| > \frac{\gamma}{n_i}
\]
and it follows at once that
\[
\tilde{d} \left( \left\{ t : q(x, t) > \beta - \frac{1}{m} \right\}, 0 \right) \geq \gamma > \frac{1}{2}.
\]
From the definition of \( f_{ap}^* \) we see that \( f_{ap}^* (x) > \beta - \frac{1}{m} \). Therefore
\[
\bigcap_{m=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_f \left( \frac{1}{n}, \beta - \frac{1}{m}, \gamma \right) \subset \left\{ x : f_{ap}^* (x) > \beta - \frac{1}{m} \right\},
\]
which implies
\[
(5) \quad \bigcap_{m=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_f \left( \frac{1}{n}, \beta - \frac{1}{m}, \gamma \right) \subset \left\{ x : f_{ap}^* (x) \geq \beta \right\}.
\]

Now let \( f_{ap}^* (x) \geq \beta \). Then for each positive integer, \( m \), there is a positive integer, \( N \), such that when \( n > N \),
\[
\left| \left\{ t \in \left( 0, \frac{1}{n} \right) : q(x, t) > \beta - \frac{1}{m} \right\} \right| > \frac{\gamma}{n},
\]
Therefore
\[
x \in \bigcup_{n=k}^{\infty} A_f \left( \frac{1}{n}, \beta - \frac{1}{m}, \gamma \right)
\]
for all \( k \). This implies that
\[
x \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_f \left( \frac{1}{n}, \beta - \frac{1}{m}, \gamma \right).
\]
Since \( f_{ap}^* (x) \geq \beta \), this same reasoning holds for any positive integer, \( m \), and so
\[
(6) \quad x \in \bigcap_{m=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_f \left( \frac{1}{n}, \beta - \frac{1}{m}, \gamma \right).
\]
In light of (5) and (6) we conclude that
\[
\left\{ x : f_{ap}^* (x) \geq \beta \right\} = \bigcap_{m=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_f \left( \frac{1}{n}, \beta - \frac{1}{m}, \gamma \right).
\]
It follows similarly that

\[ \{ x : f_{ap}^+(x) \leq \beta \} = \bigcap_{m=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} B_{f} \left( \frac{1}{n}, \beta + \frac{1}{m}, \gamma \right). \]

Now we apply the lemma to see that both

\[ \{ x : f_{ap}^-(x) \geq \beta \} \quad \text{and} \quad \{ x : f_{ap}^+(x) \leq \beta \} \]

are \( G_\delta \) sets. Using the standard theorem in Goffman [1, p. 141], it follows that \( f_{ap}^+ \in \mathcal{B}_1 \).

**Theorem 2.** Let \( f \) be a measurable function which is real-valued a.e. and let \( \beta \in [-\infty, \infty] \). Then the sets

\[ \{ x : f_{ap}^+(x) > \beta \} \quad \text{and} \quad \{ x : f_{ap}^+(x) < \beta \} \]

are both \( G_{\delta_0} \) sets and the sets

\[ \{ x : f_{ap}^-(x) < \beta \} \quad \text{and} \quad \{ x : f_{ap}^-(x) > \beta \} \]

are both \( F_{\delta_0} \) sets.

**Proof.** As before we may assume that \( f \) is real-valued everywhere. First, assume \( \beta \in \mathbb{R} \). Let \( \gamma \in (0, 1) \) and

\[ x \in \bigcap_{k=1}^{\infty} \bigcup_{0 < a < 1/k} A_f(\alpha, \beta, \gamma). \]

Then for each positive integer, \( k \), there is an \( \alpha_k \in (0, \frac{1}{k}) \) such that

\[ \left| \{ t \in (0, \alpha_k) : q(x, t) > \beta \} \right| > \alpha_k \gamma. \]

Therefore

\[ \tilde{d} \left( \{ t : q(x, t) > \beta \}, 0 \right) \geq \gamma \]

and we conclude that \( f_{ap}^+(x) > \beta \). From this it follows that if

\[ x \in \bigcup_{n=2}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{0 < a < 1/k} A_f(\alpha, \beta, \frac{1}{n}), \]

then \( f_{ap}^+(x) > \beta \). Thus

\[ (7) \quad \bigcup_{n=2}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{0 < a < 1/k} A_f(\alpha, \beta, \frac{1}{n}) \subset \{ x : f_{ap}^+(x) > \beta \}. \]

Now, suppose \( f_{ap}^+(x) > \beta \). Then there is a positive integer, \( n_0 \), such that

\[ \tilde{d} \left( \{ t : q(x, t) > \beta \}, 0 \right) > \frac{1}{n_0}. \]

This implies that for each positive integer, \( k \), there is an \( \alpha_k \in (0, \frac{1}{k}) \) such that

\[ x \in A_f(\alpha_k, \beta, \frac{1}{n_0}). \]

Therefore

\[ x \in \bigcap_{k=1}^{\infty} \bigcup_{0 < a < 1/k} A_f(\alpha, \beta, \frac{1}{n_0}) \subset \bigcup_{n=2}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{0 < a < 1/k} A_f(\alpha, \beta, \frac{1}{n}). \]
and so

\[(8) \quad \{ x : \hat{f}_{ap}^s(x) > \beta \} \subset \bigcup_{n=2}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{0 < \alpha < 1/k} A_f(\alpha, \beta, \frac{1}{n}) \].

From (7) and (8) it follows that

\[(\alpha : \hat{f}_{ap}^s(x) > \beta) = \bigcup_{n=2}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{0 < \alpha < 1/k} A_f(\alpha, \beta, \frac{1}{n}) \).

It follows from the lemma that \{ x : \hat{f}_{ap}^s(x) > \beta \} is a \( G_{\delta_0} \) set.

If \( |\beta| = \infty \), then

\[ \{ x : \hat{f}_{ap}^s(x) > \infty \} = \emptyset \quad \text{and} \quad \{ x : \hat{f}_{ap}^s(x) > -\infty \} = \bigcup_{n=1}^{\infty} \{ x : \hat{f}_{ap}^s(x) > -n \} \]

are both \( G_{\delta_0} \) sets. Note that if \( \beta \in \mathbb{R} \), then

\[ \{ x : \hat{f}_{ap}^s(x) \geq \beta \} = \bigcap_{n=1}^{\infty} \{ x : \hat{f}_{ap}^s(x) > \beta - \frac{1}{n} \} \]

\[ \{ x : \hat{f}_{ap}^s(x) \geq \infty \} = \bigcap_{n=1}^{\infty} \{ x : \hat{f}_{ap}^s(x) > n \} \]

and

\[ \{ x : \hat{f}_{ap}^s(x) \geq -\infty \} = \mathbb{R}, \]

where all three sets are \( G_{\delta_0} \) sets. Therefore, for any \( \beta \in [-\infty, \infty] \),

\[ \{ x : \hat{f}_{ap}^s(x) < \beta \} = \mathbb{R} - \{ x : \hat{f}_{ap}^s(x) \geq \beta \} \]

is an \( F_{\sigma \delta} \) set.

To prove the theorem for the lower derivate, it suffices to note that if \( g = -f \), then \( f_{ap}^s = -(\hat{g}_{ap}^s) \).

According to Goffman [1, p. 141], Theorem 2 immediately implies the following corollary.

**Corollary 3.** Let \( f \) be a measurable function which is finite a.e. Then \( \hat{f}_{ap}^s \) and \( f_{ap}^s \) are both in \( \mathcal{G}_1 \).

Corollary 3 is an improvement on a theorem of Kundu [2], who showed that if \( f \) is continuous, then \( \hat{f}_{ap}^s \) is measurable. It was shown by Preiss [4] that all ordinary approximate derivatives are in \( \mathcal{G}_1 \), and Larson [3] showed that all ordinary symmetric derivatives are in \( \mathcal{G}_1 \); Theorem 1 is a partial extension of both of these results.

**References**