ON MAXIMAL IDEALS DEPENDING ON SOME THIN SETS IN M(G)

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Abstract Let M(G) be the convolution measure algebra on the LCA group G with dual Γ, and Δ the maximal ideal space of M(G). For E ⊆ G a compact set, let Gp(E) be the subgroup of G generated algebraically by E, Runtime(E) the measures which are carried by a countable union of translates of Gp(E), and Ïƒ the natural projection from M(G) onto Runtime(E). Also let h_E be the multiplicative linear functional μ → (̂μ(1)) on M(G). Then we prove that if G is an I-group, and E an H₁-set, we get h_E ∈ Γ (i.e. the closure of Γ in Δ).

1. Introduction. Let G be a nondiscrete LCA group with dual Γ, and M(G) the convolution measure algebra of G (cf. [8, 14]). We denote by Δ = Δ M(G) the maximal ideal space of M(G). Given a Borel set E in G, let I(E) be the set of those measures μ in M(G) which satisfy \(| μ |(Gp(E) + x) = 0 for all x ∈ G, where Gp(E) is the algebraic group generated by E, and Runtime(E) = I(E)- be the set of those measures in M(G) which are singular with respect to all members of I(E). Thus I(E) and Runtime(E) are an L-ideal and an L-subalgebra of M(G), respectively, and M(G) can be decomposed into the direct sum of I(E) and Runtime(E). Moreover, each measure in Runtime(E) is carried by countable union of translates of Gp(E). Let P_E denote the natural projection from M(G) onto Runtime(E). Then P_E is multiplicative and the linear functional μ → (P_Êμ)(1) = (P_Eμ)(G) is a complex homomorphism of M(G), which we will denote by h_E.

C. F. Dunkl and D. E. Ramirez [3] proved h_E ∈ Γ (i.e. the closure of Γ in Δ) for some Borel set E. In this paper we shall study h_E for some thin sets E.

In order to prove our results, we need some notations. For μ ∈ M(G), we denote ̂μ the Gelfand transform of μ which equals to Fourier-Stieltjes transform on Γ, and \(\| ̂μ \|_∞ = \sup \{| ̂μ(γ) | : γ ∈ Γ\} \). Given a set K in G and a natural number n, we define nK = K + ... + K (n times). Also we define nK = \{0\} if n = 0, and nK = (−n)K if n is a negative integer. Given a subgroup H of G, we shall define that K is H-independent if (a) K ∩ H = ∅ , and if (b) whenever x₁, ..., x_n are finitely many distinct elements of K, (p₁, ..., p_n) ∈ Z^n, and p₁x₁ + ... + p_nx_n ∈ H, then p_jx_j ∈ H for all j = 1, 2, ..., n. Notice that when H = \{0\}, the above definition agrees with the usual definition (cf. [8]). We shall say that a LCA group G is an I-group if every nonempty open set in G contains an element of infinite order. The author wishes to thank Professor S. Saeki for his many helpful criticisms and suggestions.
2. Maximal ideals and thin sets. Let \( G \) be a LCA group and \( K \) a compact subset of \( G \). \( K \) is called an \( H_{\alpha} \)-set if \( \inf\{ \| \mu \|_{\infty} : \| \mu \| = 1, \mu \in M(K) \} = \alpha \) (\( 0 < \alpha \leq 1 \)). We shall simply call \( K \) a Helson set, if \( K \) is an \( H_{\alpha} \)-set for some \( \alpha \).

**Theorem 1.** Let \( G \) be a nondiscrete LCA I-group. Then if \( K \) is an \( H_{\alpha} \)-set, we have \( h_K \in \Gamma \). Also there exists \( K \) and \( H_{\alpha} \)-set (\( 0 < \alpha < 1 \)) such that \( h_K \notin \Gamma \).

**Theorem 2.** Let \( G \) be a nondiscrete LCA group. Then there exists \( K \) a non-Helson set in \( G \) such that
\[
\lim_{\gamma \to \infty} \| \gamma - 1 \|_K = 0 \quad \text{and} \quad h_K \notin \Gamma.
\]

**Corollary 1** (cf. [1]). Let \( G \) be a nondiscrete LCA I-group. Then there exists \( \{ h_K \} \) uncountable maximal ideals such that \( h_K \in \Delta \setminus \Gamma \), and there exist \( \{ h_K \} \) uncountable maximal ideals such that \( h_K \in \Gamma \setminus \Gamma \).

**Corollary 2.** Under the notation of Theorem 2, \( h_K \) is not a strong boundary point for the uniform closure of \( M(G) \) in \( C(\Delta) \).

The next result is obtained by C. F. Dunkl and D. E. Ramirez [2].

**Proposition.** Let \( f \) be an element in \( \Delta \). Then \( f \) is contained in \( \Gamma \) if and only if
\[
| \hat{f}(\gamma) | \leq \| f \|_{\infty} \quad \text{for all} \quad \gamma \in \Gamma.
\]

In order to prove our results, we need some lemmas.

**Lemma 1.** Let \( K \) be an \( H_{\alpha} \)-set with \( K \ni 0 \) in \( G \), \( \epsilon \) a positive number, and \( E \) a compact subset in \( G \) with \( E \cap Gp(K) = \emptyset \). Then there exists a probability measure \( \mu_{\epsilon,E} \in M(G) \) such that \( \| \hat{\mu}_{\epsilon,E} - 1 \|_K < \epsilon \) and \( \| \hat{\mu}_{\epsilon,E} \|_E < \epsilon \). Moreover, if we put \( A = \{ (\epsilon, E) \mid \epsilon > 0, \text{and } E \text{ is a compact set with } E \cap Gp(K) = \emptyset \} \), and we define \( (\epsilon_1, E_1) \prec (\epsilon_2, E_2) \) for \( (\epsilon, E) \in A \) (\( i = 1, 2 \)), if \( \epsilon_2 < \epsilon_1 \) and \( E_2 \supset E_1 \), we obtain that \( \lim_{\gamma \to \infty} \mu_{\epsilon_1,E_1} = m_\Lambda \) in the \( w^* \)-topology of \( M(\Gamma_\Lambda) \), where \( \Gamma_\Lambda \) is the Bohr compactification of \( \Gamma \), \( \Lambda \) the annihilator of \( Gp(K) \) in \( \Gamma_\Lambda \), and \( m_\Lambda \) the normalized Haar measure of \( \Lambda \).

**Proof.** The first half is obtained by modifying [9, Theorem 5] (cf. [15]). We remark that \( \{ \mu_{\epsilon,E} \} \) is a net in \( M(G) \). Then we have the last half by the first half. Q.E.D.

**Lemma 2.** Let \( K \) be an \( H_{\alpha} \)-set containing \( 0 \) in \( G \), \( \epsilon \) a positive number, and \( F_0 \) a finite subset of \( G \) which is \( Gp(K) \)-independent. Also let \( E_0 \) be a compact set in \( G \) with \( E_0 \cap Gp(F_0 \cup K) = \emptyset \). Then there exists \( \nu \) a probability measure in \( M(\Gamma) \) such that
\[
| \hat{\nu} - 1 | < \epsilon \quad \text{on } F_0 \cup K \quad \text{and} \quad | \hat{\nu} | < \epsilon \quad \text{on } E_0.
\]

**Proof.** Let \( m_\Lambda \) be as in Lemma 1. By [8], for \( \eta > 0 \), there exists \( P_0 \) a trigonometric polynomial on \( \Lambda \) such that
\begin{enumerate}
\item \( P_0 \geq 0 \) on \( \Lambda \) and \( \| P_0 m_\Lambda \| = 1 \),
\item \( \text{supp} \hat{P}_0 \subseteq Gp(F_0 \cup Gp(K)) \),
\item \( |(P_0 m_\Lambda) - 1| < \eta \) on \( F_0 + Gp(K) \), and
\item \( |(P_0 m_\Lambda)| < \eta \) on \( E_0 + Gp(K) \).
\end{enumerate}
Choose a sufficiently large finite set $F$ with $\{0\} \cup F_0 \subset F \subset \text{Gp}(F_0)$ and $(F - F) \cap \text{Gp}(K) = \{0\}$, and set $P(\gamma) = \sum_{x \in F} C_x \gamma(x)$ on $\Gamma$, where $C_x = \hat{P}_0(x + \text{Gp}(K)) (x \in F)$. For $\mu_{e,E}$ as in Lemma 1, we define $\nu_{e,E} = |P| \mu_{e,E}$. Then by $\Gamma \subset \Gamma_0$ and Lemma 1, we get that

(5)' $\|\nu_{e,E} - P\mu_{e,E}\| \rightarrow 0$ as $(e, E) \rightarrow \infty$, and

(6)' $\|\nu_{e,E} - P\mu_{e,E}\| = \|\nu_{e,E} - P\| \rightarrow 0$ as $(e, E) \rightarrow \infty$.

(7)' $\|\nu_{e,E} - P\mu_{e,E}\| \rightarrow 0$ as $(e, E) \rightarrow \infty$.

For the above trigonometric polynomial $P$ and for $\eta > 0$, by (5), (6), the assumption of Lemmas 2 and 1, there exists $(e, E) \in A$ such that

(7) $E \supset ((x_1 - x_2 | x_1, x_2 \in F, x_1 \neq x_2) + K) \cup (E_0 - F)$,

(8) $E \subset \text{Gp}(K) = \emptyset$,

(9) $\|\hat{\mu}_{e,E}\| < \eta / (\sum_{x \in F} |C_x| + 1)$,

(10) $\|\nu_{e,E}\| < \eta$, and

(11) $\|\nu_{e,E} - P\mu_{e,E}\| < \eta$.

If we define $\nu = \nu_{e,E}/\|\nu_{e,E}\|$ in $M(\Gamma)$, $\nu$ will satisfy the properties of Lemma 2. In fact, for $y \in F_0$ and $k \in K \cup \{0\}$, by the definition of $\nu$, we have that $|\hat{\nu}(y + k) - 1| \leq (|\hat{\nu}_{e,E}(y + k) - 1| + 1 - \|\nu_{e,E}\|)/\|\nu_{e,E}\|$. By (7), (8), (9), (10), and (11), we obtain

\[
|\hat{\nu}_{e,E}(y + k) - 1| \leq \eta + \|P\mu_{e,E}\| \leq \eta + \|P\| \sum_{x \neq y} |C_x| \|\hat{\nu}_{e,E}(y + k - x)|
\]

\[
\leq \eta + 2\eta + \eta,
\]

and $|\hat{\nu}(y + k) - 1| \leq (5\eta)/(1 - \eta)$. Also for $e \in E_0$, we have that

$|\hat{\nu}(e)| \leq (\eta + |(P\mu_{e,E})^\gamma(e)|)/\|\nu_{e,E}\|$

and $|\hat{\nu}(e)| \leq (2\eta)/(1 - \eta)$ by (7) and (8). Thus $\nu$ satisfies what we want for sufficiently small $\eta$. Q.E.D.

**Proof of Theorem 1.** Let $\mu$ be any measure in $M(G)$. By $|\hat{\mu}(h_K)| \leq |(P\mu)^\gamma(1)|$ and the Proposition, it will suffice to show $\|(P\mu\mu)^\gamma\| \leq \|\hat{\mu}\|_\infty$ for all $\mu \in M(G)$. Let $K > 0$, there exists $F$ a finite set such that

(1) $\|\nu_0 - P\mu\| < \varepsilon$, where $\nu_0 = \mu|_{\text{Gp}(K) + F}$.

Let $k \in K$ be fixed. Since $K$ is an $H_1$-set, $K - \{k\}$ is an $H_1$-set with $K - \{k\} \ni 0$ in $G$. Then there exists $F_0$ a finite set in $G$ such that $F_0$ is $\text{Gp}(K - \{k\})$-independent set and $\text{Gp}(F \cup K) = \text{Gp}(F_0 \cup (K - \{k\}))$. So there exists $N$ a natural number such that

(2) $\|\tau - \nu_0\| < \varepsilon$, where $\tau = \mu|_{K_N}$, and

$K_N = N(F_0 \cup (K - \{k\}) \cup (-F_0) \cup (-K + \{k\}))$.  

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Now let $E_0$ be a compact subset of $G$ such that
\[(3) \| (\mu - \tau) \|_{G, E_0} < \varepsilon \text{ and } E_0 \cap \text{Gp}(F_0 \cup (K - \{ k \})) = \emptyset.\]
By Lemma 2, there exists $\nu$ a probability measure in $M(G)$ such that
\[(4) \| \hat{\nu} - 1 \|_{F_0 \cup (K - \{ k \})} < \varepsilon/N, \text{ and} \]
\[(5) \| \hat{\nu} \|_{F_0} < \varepsilon.\]
By (1), (2), and (3), we get that
\[\int_G \hat{\nu} (x) \gamma(x) \, d\mu(x) = \int_{K_N} \hat{\nu} (x) \gamma(x) \, d\mu(x) \geq \int_{F_0} \hat{\nu} (x) \gamma(x) \, d\mu(x) - 3\varepsilon.\]
Also by (4) and (5), we have that $| \hat{\nu} (x) - 1 | < \varepsilon$ for $x \in K_N$, and
\[\int_G \hat{\nu} (x) \gamma(x) \, d\mu(x) < | \hat{\gamma} \gamma | - \varepsilon - \varepsilon - 3\varepsilon.\]
On the other hand, we have that $| \int_G \hat{\nu} (x) \gamma(x) \, d\mu(x) | \leq \| \hat{\nu} \|_{\infty} \| \nu \| = \| \hat{\nu} \|_{\infty}$. Then we get that $| \hat{\gamma} \gamma | \leq \| \hat{\nu} \|_{\infty} + 5\varepsilon$. Thus by (1) and (2), we have that $\| (P_k \mu)(\gamma) \| \leq \| \hat{\mu} \|_{\infty} + 7\varepsilon$ and $\| (P_k \mu) \|_{\infty} \leq \| \hat{\mu} \|_{\infty}$.

The last half is as follows. By Körner [6], there exist $K$ a perfect independent $H_\alpha$-set ($0 < \alpha < 1$) and a probability measure $\sigma \in M(K)$ such that $\lim_{\gamma \to \infty} | \hat{\sigma}(\gamma) | < 1$. If we assume $h_K \in \hat{\Gamma}$, there exist $\{ \gamma_n \}$ such that $\lim_{n \to \infty} \int \gamma_n \, d\sigma = \int h_K \, d\sigma$. Since we have that
\[\lim_{n \to \infty} \int \gamma_n \, d\sigma = \lim_{\gamma \to \infty} | \sigma(\gamma) | < 1\]
and $\int h_K \, d\sigma = 1$ by the property of $\sigma$, we get the contradiction. Q.E.D.

**Lemma 3** [7, p. 114]. Let $G$ be an nondiscrete LCA group, and $L$ a totally disconnected perfect independent set and non-Helson set in $G$. Then there exists $K \subset L$ a clopen non-Helson set in the relative topology of $L$ such that $\| \hat{\mu} \|_{\infty} \geq 3 \| \hat{\mu} \|_{\infty}$ for some nonzero measure $\mu$ in $M(L)$.

**Lemma 4** [9]. Let $G$ be a nondiscrete LCA group, $F \subset G$ a finite set, and $E \subset G$ a closed subset with $E \cap \text{Gp}(F) = \emptyset$. Then to each $\varepsilon > 0$, there exists $f \in A(G) = L^1(\Gamma)$ such that $0 \leq f \leq 1$ on $G$, $f = 1$ on $F$, $f = 0$ on some neighborhood of $E$, and $\| f \|_{A(G)} < 1 + \varepsilon$.

**Proof of Theorem 2.** Since $G$ is a nondiscrete LCA group, there exists $L$ a totally disconnected perfect independent set and non-Helson set such that
\[\lim_{\gamma \to \infty} \| \gamma - 1 \|_L = 0\]
(cf. [5, 8]). Then by Lemma 3, there exists $K \subset L$ a perfect subset with the property of Lemma 3. Now we shall show $h_K \notin \hat{\Gamma}$. By the Proposition, it is sufficient to show that we get a contradiction if we assume $\| (P_k v) \|_{\infty} \| \hat{v} \|_{\infty}$ for all $v \in M(G)$. We define $K' = L \setminus K$, and for $\mu$ as in Lemma 3, $P_k \mu = \mu_1 + \mu_2$ where $\mu_1 = \mu_K$ and $\mu_2 = \mu_{K'}$. Since $L$ is an independent set, we have that (a) $\mu_1 = \mu_{\text{Gp}(K)}$, and (b) $\mu_2 \in M_d(K')$. By (a) and the choice of $\mu$, we get $\| \hat{\mu}_1 \|_{\infty} = \| \hat{\mu}_{\text{Gp}(K)} \|_{\infty} \geq 3 \| \hat{\mu} \|_{\infty}$. 

Also by (b), we assume that the support of $\mu_2$ is a finite set. By Lemma 4, to each $\epsilon > 0$, there exists $f \in A(G)$ ($0 \leq f \leq 1$) such that $f = 1$ on $\text{supp} \mu_2$, $f = 0$ on $K$, and $\|f\|_{A(G)} < 1 + \epsilon$. Then for $\gamma \in \Gamma$, we have that

$$\int_G \gamma f d(P_K \mu) = \int_K \gamma f d(P_K \mu) + \int_K' \gamma f d(P_K \mu) = \int_K' \gamma f d(P_K \mu).$$

Thus we get that

$$\left| \int_G \gamma d(\mu_2) \right| = \left| \int_K \gamma f d(P_K \mu) \right| = \left| \int_G \gamma f d(P_K \mu) \right| \leq \|f\|_{A(G)} \|\mu_2\|_\infty < (1 + \epsilon) \|\mu_2\|_\infty.$$ 

and $\|\mu_2\|_\infty \leq \|\mu_2\|_\infty$ ($\leq \|\mu\|_\infty$). On the other hand, by the assumption of the contradiction and the property of Lemma 3, we have that

$$\|\hat{\mu}\|_\infty \geq \|\hat{\mu}_1\|_\infty - \|\hat{\mu}_2\|_\infty = \|\hat{\mu}_{Gp(K)}\|_\infty - \|\hat{\mu}_2\|_\infty \geq 3\|\hat{\mu}\|_\infty - \|\mu\|_\infty = 2\|\hat{\mu}\|_\infty.$$ 

Therefore we get a required contradiction. Q.E.D.

Proof of Corollary 1. By the well-known fact [7], there exists $K$ a totally disconnected perfect independent set with $\sigma(0) \in M(K)$ and $\sigma(\gamma) \to 0$ as $\gamma \to \infty$. Then $K$ is divided with $K = K_1 \cup K_2$ such that $K_1 \cap K_2 \neq \emptyset$ and $K_1, K_2$ are nonempty perfect compact subsets with $\tau(0) \in M(K_1)$ and $\tau(\gamma) \to 0$ as $\gamma \to \infty$. Since $K$ is an independent set, $(K_1 + \{x\})_{x \in K}$, are pairwise disjoint sets. By [11], there exists $K(x)$ a totally disconnected perfect independent and non-Helson set with $K(x) \subset K_1 + \{x\}$. Then $(K(x))_{x \in K_1, x \in K_2}$ are a countable set, and $\{h_{K(x)}\}_{x \in K_2}$, are what we require. In fact, we assume $x \neq y$ and $K(x) \cap (Gp(K(y)) + z) \neq \emptyset$. Hence, if there exist $w, w' \in K(x)$ such that $w = z + \sum n_k(y, i)$ and $w' = z + \sum n'_k(y, i)$ where $n_k, n'_k$ integers and $k(y, i), k'(y, i) \in K(y)$, we have $w - w' = -\sum n_k(y, i) + \sum n'_k(y, i)$ since we have $Gp(K(x)) \cap Gp(K(y)) = \{0\}$ by $x \neq y$, we get $w = w'$.

Thus there exists $\mu$ a continuous probability measure on $K(x)$ such that $\hat{\mu}(h_{K(x)}) = 0$. Therefore we proved the first half. Also it is well known that there exists $K$ a totally disconnected perfect independent $H_1$-set in $G$. So the last half can be proved in the same way as the above method by Theorem 2. Q.E.D.

The proof of Corollary 2 is clear by Theorem 2 and [13].

3. $H$-Kronecker set. In this section, we show a generalization of [12] in some sense. First we define $H$-Kronecker set.

Definition. Let $G$ be a nondiscrete LCA group, $K$ a compact set, and $H$ a closed subgroup of $G$. $K$ is called an $H$-Kronecker set if, to each $\epsilon > 0$ and $f \in C(K)$ with $|f| = 1$, there exists $\gamma \in H^\perp$ such that $\|f - \gamma\|_{K} < \epsilon$. We simply call $K$ a Kronecker set, if $H = \{0\}$.

Next we shall show the existence of a nontrivial $H$-Kronecker set.
Theorem 3. Let $G$ be a metric $I$-group, and $H$ a $\sigma$-compact closed subgroup with Haar measure zero such that $\{px: x \in U\} \not\subset H$ for any natural number $p$ and for any neighborhood $U$ of $0 \in G$. Then there exists $K$ an $H$-Kronecker set which is a totally disconnected perfect set.

The proof is obtained by modifying [8, 5.2].

Theorem 4. Under the above assumption, let $\rho$ be a nonzero continuous measure on $K$. Then we have that for any $\gamma \in \Gamma$, $\mu \in M(G)$ $(1 \leq i \leq n)$, and $\varepsilon > 0$, there exists $\gamma_0 \in \Gamma$ such that $|(h_{\mu^i})(\gamma) - (h_{\mu^i})(\gamma_0)| < \varepsilon$ $(1 \leq i \leq n)$ and $|\hat{\rho}(\gamma_0)| > \|\rho\|/2$.

Proof. Let $\pi$ be the natural mapping from $G$ to $G/H$, and $\hat{\pi}: M(G) \to M(G/H)$ the mapping induced by $\pi$. Then we remark $\hat{\pi}(\mu) = \hat{\mu}|_{H^\perp}$ for all $\mu \in M(G)$. Now by the definition of $K$, $\pi(K)$ is a Kronecker set on $G/H$. Hence, by an application of [12] in $G/H$, there exists $\gamma_0 \in H^\perp$ such that

$$|\hat{\pi}(\hat{\chi}_{\mu^i})(\gamma) - \hat{\pi}(\hat{\chi}_{\mu^i})(\gamma_0)| < \varepsilon \quad (1 \leq i \leq n),$$

and $|\hat{\pi}(\hat{\rho})(\gamma_0)| > \|\rho\|/2$ (cf. [4]).

That is, we have

$$|(h_{\mu^i})(\gamma) - (h_{\mu^i})(\gamma_0 + \gamma)| < \varepsilon \quad (1 \leq i \leq n),$$

and $|\hat{\rho}(\gamma_0 + \gamma)| > \|\rho\|/2$. This establishes the result. Q.E.D.

References