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ON MAXIMAL IDEALS DEPENDING ON SOME THIN SETS IN $M(G)$

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ABSTRACT. Let $M(G)$ be the convolution measure algebra on the LCA group G with dual Γ, and Δ the maximal ideal space of $M(G)$. For $E \subset G$ a compact set, let $Gp(E)$ be the subgroup of G generated algebraically by E, $R(E)$ the measures which are carried by a countable union of translates of $Gp(E)$, and P_E the natural projection from $M(G)$ onto $R(E)$. Also let h_E be the multiplicative linear functional $\mu \mapsto (P_E\mu)\hat{}(1)$ on $M(G)$. Then we prove that if G is an I-group, and E an H_1-set, we get $h_E \in \overline{\Gamma}$ (i.e. the closure of Γ in Δ).

1. Introduction. Let G be a nondiscrete LCA group with dual Γ, and $M(G)$ the convolution measure algebra of G (cf. [**8**, **14**]). We denote by $\Delta = \Delta_{M(G)}$ the maximal ideal space of $M(G)$. Given a Borel set E in G, let $I(E)$ be the set of those measures μ in $M(G)$ which satisfy $|\mu|(Gp(E) + x) = 0$ for all $x \in G$, where $Gp(E)$ is the algebraic group generated by E, and $R(E) = I(E)^{\perp}$ be the set of those measures in $M(G)$ which are singular with respect to all members of $I(E)$. Thus $I(E)$ and $R(E)$ are an L-ideal and an L-subalgebra of $M(G)$, respectively, and $M(G)$ can be decomposed into the direct sum of $I(E)$ and $R(E)$. Moreover, each measure in $R(E)$ is carried by countable union of translates of $Gp(E)$. Let P_E denote the natural projection from $M(G)$ onto $R(E)$. Then P_E is multiplicative and the linear functional $\mu \mapsto (P_E\mu)\hat{}(1) = (P_E\mu)(G)$ is a complex homomorphism of $M(G)$, which we will denote by h_E.

C. F. Dunkl and D. E. Ramirez [3] proved $h_E \in \overline{\Gamma}$ (i.e. the closure of Γ in Δ) for some Borel set E. In this paper we shall study h_E for some thin sets E.

In order to prove our results, we need some notations. For $\mu \in M(G)$, we denote $\hat{\mu}$ the Gelfand transform of μ which equals to Fourier-Stieltjes transform on Γ, and $\|\hat{\mu}\|_\infty = \sup\{|\hat{\mu}(\gamma)| : \gamma \in \Gamma\}$. Given a set K in G and a natural number n, we define $nK = K + \cdots + K$ (n times). Also we define $nK = \{0\}$ if $n = 0$, and $nK = (-n)(-K)$ if n is a negative integer. Given a subgroup H of G, we shall define that K is H-independent if (a) $K \cap H = \varnothing$, and if (b) whenever $x_1, \ldots, x_n$ are finitely many distinct elements of K, $(p_1, \ldots, p_n) \in \mathbf{Z}^n$, and $p_1x_1 + \cdots + p_nx_n \in H$, then $p_jx_j \in H$ for all $j = 1, 2, \ldots, n$. Notice that when $H = \{0\}$, the above definition agrees with the usual definition (cf. [**8**]). We shall say that a LCA group G is an I-group if every nonempty open set in G contains an element of inifinte order. The author wishes to thank Professor S. Saeki for his many helpful criticisms and suggestions.

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2. Maximal ideals and thin sets. Let G be a LCA group and K a compact subset of G. K is called an H_α-set if $\inf\{\|\hat{\mu}\|_\infty \mid \|\mu\| = 1, \mu \in M(K)\} = \alpha$ $(0 < \alpha \leq 1)$. We shall simply call K a Helson set, if K is an H_α-set for some α.

THEOREM 1. *Let G be a nondiscrete LCA I-group. Then if K is an H_1-set, we have $h_k \in \bar{\Gamma}$. Also there exists K and H_α-set $(0 < \alpha < 1)$ such that $h_K \notin \bar{\Gamma}$.*

THEOREM 2. *Let G be a nondiscrete LCA group. Then there exists K a non-Helson set in G such that*

$$\varliminf_{\substack{\gamma\to\infty\\ \gamma\in\Gamma}} \|\gamma - 1\|_K = 0 \quad \textit{and} \quad h_K \notin \bar{\Gamma}.$$

COROLLARY 1 (cf. [1]). *Let G be a nondiscrete LCA I-group. Then there exists $\{h_K\}$ uncountable maximal ideals such that $h_K \in \Delta \setminus \bar{\Gamma}$, and there exist $\{h_K\}$ uncountable maximal ideals such that $h_K \in \bar{\Gamma} \setminus \Gamma$.*

COROLLARY 2. *Under the notation of Theorem 2, h_K is not a strong boundary point for the uniform closure of $M(G)\hat{}$ in $C(\Delta)$.*

The next result is obtained by C. F. Dunkl and D. E. Ramirez [2].

PROPOSITION. *Let f be an element in Δ. Then f is contained in $\bar{\Gamma}$ if and only if $|\hat{\mu}(f)| \leq \|\hat{\mu}\|_\infty$ for all $\mu \in M(G)$.*

In order to prove our results, we need some lemmas.

LEMMA 1. *Let K be an H_1-set with $K \ni 0$ in G, ε a positive number, and E a compact subset in G with $E \cap Gp(K) = \varnothing$. Then there exists a probability measure $\mu_{\varepsilon,E} \in M(G)$ such that $\|\hat{\mu}_{\varepsilon,E} - 1\|_K < \varepsilon$ and $\|\hat{\mu}_{\varepsilon,E}\|_E < \varepsilon$. Moreover, if we put $A = \{(\varepsilon, E) \mid \varepsilon > 0$, and E is a compact set with $E \cap Gp(K) = \varnothing\}$, and we define $(\varepsilon_1, E_1) \prec (\varepsilon_2, E_2)$ for $(\varepsilon_i, E_i) \in A$ $(i = 1,2)$, if $\varepsilon_2 < \varepsilon_1$ and $E_2 \supset E_1$, we obtain that $\lim \mu_{\varepsilon,E} = m_\Lambda$ in the w^*-topology of $M(\Gamma_b)$, where Γ_b is the Bohr compactification of Γ, Λ the annihilator of $Gp(K)$ in Γ_b, and m_Λ the normalized Haar measure of Λ.*

PROOF. The first half is obtained by modifying [9, Theorem 5] (cf. [15]). We remark that $\{\mu_{\varepsilon,E}\}_A$ is a net in $M(G)$. Then we have the last half by the first half. Q.E.D.

LEMMA 2. *Let K be an H_1-set containing 0 in G, ε a positive number, and F_0 a finite subset of G which is $Gp(K)$-independent. Also let E_0 be a compact set in G with $E_0 \cap Gp(F_0 \cup K) = \varnothing$. Then there exists ν a probability measure in $M(\Gamma)$ such that $|\hat{\nu} - 1| < \varepsilon$ on $F_0 \cup K$ and $|\hat{\nu}| < \varepsilon$ on E_0.*

PROOF. Let m_Λ be as in Lemma 1. By [8], for $\eta > 0$, there exists P_0 a trigonometric polynomial on Λ such that

(1) $P_0 \geq 0$ on Λ and $\|P_0 m_\Lambda\| = 1$,
(2) supp $\hat{P}_0 \subset Gp[F_0 \cup Gp(K)]$,
(3) $|(P_0 m_\Lambda)\hat{} - 1| < \eta$ on $F_0 + Gp(K)$, and
(4) $|(P_0 m_\Lambda)\hat{}| < \eta$ on $E_0 + Gp(K)$.

Choose a sufficiently large finite set F with $\{0\} \cup F_0 \subset F \subset Gp(F_0)$ and $(F - F) \cap Gp(K) = \{0\}$, and set $P(\gamma) = \Sigma_{x \in F} C_x x(\gamma)$ on Γ, where $C_x = \hat{P}_0(x + Gp(K))$ $(x \in F)$. For $\mu_{\varepsilon,E}$ as in Lemma 1, we define $\nu_{\varepsilon,E} = |P|\mu_{\varepsilon,E}$. Then by $\Gamma \subset \Gamma_b$ and Lemma 1, we get that

(5)′ $\|\nu_{\varepsilon,E}\| = \int_{\Gamma_b} |P|\, d\mu_{\varepsilon,E}$,

(6)′ $|\int_{\Gamma_b} |P|\, d\mu_{\varepsilon,E} - \int_{\Gamma_b} |P|\, dm_\Lambda | \mapsto 0$ as $(\varepsilon, E) \mapsto \infty$, and

(7)′ $\|\nu_{\varepsilon,E} - P\mu_{\varepsilon,E}\| = \int_{\Gamma_b} ||P| - P|\, d\mu_{\varepsilon,E} \mapsto \int_{\Gamma_b} ||P| - P|\, dm_\Lambda$ as $(\varepsilon, E) \mapsto \infty$.

On the other hand, by the definition of P, we have $\int_{\Gamma_b} |P|\, dm_\Lambda = 1$ and $P = |P|$ on Λ. So by (5)′, (6)′ and (7)′, we get that

(5) $\|\nu_{\varepsilon,E}\| \mapsto 1$ as $(\varepsilon, E) \mapsto \infty$, and

(6) $\|\nu_{\varepsilon,E} - P\mu_{\varepsilon,E}\| \mapsto 0$ as $(\varepsilon, E) \mapsto \infty$.

For the above trigonometric polynomial P and for $\eta > 0$, by (5), (6), the assumption of Lemmas 2 and 1, there exists $(\varepsilon, E) \in A$ such that

(7) $E \supset (\{x_1 - x_2 \mid x_1, x_2 \in F, x_1 \neq x_2\} + K) \cup (E_0 - F)$,

(8) $E \subset Gp(K) = \varnothing$,

(9) $\|\hat{\mu}_{\varepsilon,E}\|_E < \eta/(\Sigma_{x \in F} |C_x| + 1)$,

(10) $|\,\|\nu_{\varepsilon,E}\| - 1| < \eta$, and

(11) $\|\nu_{\varepsilon,E} - P\mu_{\varepsilon,E}\| < \eta$.

If we define $\nu = \nu_{\varepsilon,E}/\|\nu_{\varepsilon,E}\|$ in $M(\Gamma)$, ν will satisfy the properties of Lemma 2. In fact, for $y \in F_0$ and $k \in K \cup \{0\}$, by the definition of ν, we have that $|\hat{\nu}(y + k) - 1| \leqslant (|\hat{\nu}_{\varepsilon,E}(y + k) - 1| + |1 - \|\nu_{\varepsilon,E}\||)/\|\nu_{\varepsilon,E}\|$. By (7), (8), (9), (10), and (11), we obtain

$$
\begin{aligned}
|\hat{\nu}_{\varepsilon,E}(y + k) - 1| &\leqslant \eta + |(P\mu_{\varepsilon,E})\hat{\ }(y + k) - 1| \\
&\leqslant \eta + |C_y \hat{\mu}_{\varepsilon,E}(k) - 1| + \sum_{\substack{x \in F \\ x \neq y}} |C_x|\,|\hat{\mu}_{\varepsilon,E}(y + k - x)| \\
&\leqslant \eta + 2\eta + \eta,
\end{aligned}
$$

and $|\hat{\nu}(y + k) - 1| \leqslant (5\eta)/(1 - \eta)$. Also for $e \in E_0$, we have that

$$
|\hat{\nu}(e)| \leqslant \big(\eta + |(P\mu_{\varepsilon,E})\hat{\ }(e)|\big)/\|\nu_{\varepsilon,E}\|
$$

and $|\hat{\nu}(e)| \leqslant (2\eta)/(1 - \eta)$ by (7) and (8). Thus ν satisfies what we want for sufficiently small η. Q.E.D.

Proof of Theorem 1. Let μ be any measure in $M(G)$. By $|\hat{\mu}(h_K)| = |(P_K\mu)\hat{\ }(1)|$ and the Proposition, it will suffice to show $\|(P_K\mu)\hat{\ }\|_\infty \leqslant \|\hat{\mu}\|_\infty$ for all $\mu \in M(G)$. For $\varepsilon > 0$, there exists F a finite set such that

(1) $\|\nu_0 - P_K\mu\| < \varepsilon$, where $\nu_0 = \mu|_{Gp(K)+F}$.

Let $k \in K$ be fixed. Since K is an H_1-set, $K - \{k\}$ is an H_1-set with $K - \{k\} \ni 0$ in G. Then there exists F_0 a finite set in G such that F_0 is $Gp(K - \{k\})$-independent and $Gp(F \cup K) = GP(F_0 \cup (K - \{k\}))$. So there exists N a natural number such that

(2) $\|\tau - \nu_0\| < \varepsilon$, where $\tau = \mu|_{K_N}$, and

$$
K_N = N\big(F_0 \cup (K - \{k\}) \cup (-F_0) \cup (-K + \{k\})\big).
$$

Now let E_0 be a compact subset of G such that

(3) $\|(\mu - \tau)|_{G\setminus E_0}\| < \varepsilon$ and $E_0 \cap Gp(F_0 \cup (K - \{k\})) = \varnothing$.

By Lemma 2, there exists ν a probability measure in $M(\Gamma)$ such that

(4) $\|\hat{\nu} - 1\|_{F_0 \cup (K - \{k\})} < \varepsilon/N$, and

(5) $\|\hat{\nu}\|_{E_0} < \varepsilon$.

By (1), (2), and (3), we get that

$$\left|\int_G \hat{\nu}(x)\gamma(x)\,d\mu(x)\right| \geqslant \left|\int_{K_N} \hat{\nu}(x)\gamma(x)\,d\mu(x)\right| - \left|\int_{E_0} \hat{\nu}(x)\gamma(x)\,d\mu(x)\right| - 3\varepsilon.$$

Also by (4) and (5), we have that $|\hat{\nu}(x) - 1| < \varepsilon$ for $x \in K_N$, and

$$\left|\int_G \hat{\nu}(x)\gamma(x)\,d\mu(x)\right| < |\hat{\tau}(\gamma)| - \varepsilon - \varepsilon - 3\varepsilon.$$

On the other hand, we have that $|\int_G \hat{\nu}(x)\gamma(x)\,d\mu(x)| \leqslant \|\hat{\mu}\|_\infty \|\nu\| = \|\hat{\mu}\|_\infty$. Then we get that $|\hat{\tau}(\gamma)| \leqslant \|\hat{\mu}\|_\infty + 5\varepsilon$. Thus by (1) and (2), we have that $|(P_K\mu)\hat{}(\gamma)| \leqslant \|\hat{\mu}\|_\infty + 7\varepsilon$ and $\|(P_K\mu)\hat{}\|_\infty \leqslant \|\hat{\mu}\|_\infty$.

The last half is as follows. By Körner [**6**], there exist K a perfect independent H_α-set $(0 < \alpha < 1)$ and a probability measure $\sigma \in M(K)$ such that $\overline{\lim}_{\gamma\to\infty} |\hat{\sigma}(\gamma)| < 1$. If we assume $h_K \in \bar{\Gamma}$, there exist $\{\gamma_n\}$ such that $\lim_{n\to\infty} \int \gamma_n\,d\sigma = \int h_K\,d\sigma$. Since we have that

$$\lim_{n\to\infty} \left|\int \gamma_n\,d\sigma\right| = \overline{\lim_{\gamma\to\infty}}\, |\sigma(\gamma)| < 1$$

and $\int h_K\,d\sigma = 1$ by the property of σ, we get the contradiction. Q.E.D.

LEMMA 3 [7, p. 114]. *Let G be anondiscrete LCA group, and L a totally disconnected perfect independent set and non-Helson set in G. Then there exists $K \subset L$ a clopen non-Helson set in the relative topology of L such that $\|\hat{\mu}_{|Gp(K)}\|_\infty \geqslant 3\|\hat{\mu}\|_\infty$ for some nonzero measure μ in $M(L)$.*

LEMMA 4 [**9**]. *Let G be a nondiscrete LCA group, $F \subset G$ a finite set, and $E \subset G$ a closed subset with $E \cap Gp(F) = \varnothing$. Then to each $\varepsilon > 0$, there exists $f \in A(G) = L^1(\Gamma)\hat{}$ such that $0 \leqslant f \leqslant 1$ on G, $f = 1$ on F, $f = 0$ on some neighborhood of E, and $\|f\|_{A(G)} < 1 + \varepsilon$.*

PROOF OF THEOREM 2. Since G is a nondiscrete LCA group, there exists L a totally disconnected perfect independent set and non-Helson set such that

$$\underline{\lim}_{\substack{\gamma\to\infty \\ \gamma\in\Gamma}} \|\gamma - 1\|_L = 0$$

(cf. [**5, 8**]). Then by Lemma 3, there exists $K \subset L$ a perfect subset with the property of Lemma 3. Now we shall show $h_K \notin \bar{\Gamma}$. By the Proposition, it is sufficient to show that we get a contradiction if we assume $\|(P_K\nu)\hat{}\|_\infty \infty \|\hat{\nu}\|_\infty$ for all $\nu \in M(G)$. We define $K' = L \setminus K$, and for μ as in Lemma 3, $P_K\mu = \mu_1 + \mu_2$ where $\mu_1 = \mu_{|K}$ and $\mu_2 = \mu_{|K'}$. Since L is an independent set, we have that (a) $\mu_1 = \mu_{|Gp(K)}$, and (b) $\mu_2 \in M_d(K')$. By (a) and the choice of μ, we get $\|\hat{\mu}_1\|_\infty = \|\hat{\mu}_{|Gp(K)}\|_\infty \geqslant 3\|\hat{\mu}\|_\infty$.

Also by (b), we assume that the support of μ_2 is a finite set. By Lemma 4, to each $\varepsilon > 0$, there exists $f \in A(G)$ $(0 \leq f \leq 1)$ such that $f = 1$ on $\operatorname{supp}\mu_2$, $f = 0$ on K, and $\|f\|_{A(G)} < 1 + \varepsilon$. Then for $\gamma \in \Gamma$, we have that

$$\int_G \gamma f\, d(P_K\mu) = \int_K \gamma f\, d(P_K\mu) + \int_{K'} \gamma f\, d(P_K\mu) = \int_{K'} \gamma f\, d(P_K\mu).$$

Thus we get that

$$\left|\int_G \gamma\, d\mu_2\right| = \left|\int_{K'} \gamma f\, d(P_K\mu)\right| = \left|\int_G \gamma f\, d(P_K\mu)\right|$$
$$\leq \|f\|_{A(G)} \|(P_K\mu)\hat{}\|_\infty < (1+\varepsilon)\|(P_K\mu)\hat{}\|_\infty,$$

and $\|\hat{\mu}_2\|_\infty \leq \|(P_K\mu)\hat{}\|_\infty$ $(\leq \|\hat{\mu}\|_\infty)$. On the other hand, by the assumption of the contradiction and the property of Lemma 3, we have that

$$\|\hat{\mu}\|_\infty \geq \|\hat{\mu}_1\|_\infty - \|\hat{\mu}_2\|_\infty = \|\hat{\mu}_{|Gp(K)}\|_\infty - \|\hat{\mu}_2\|_\infty$$
$$\geq 3\|\hat{\mu}\|_\infty - \|\hat{\mu}\|_\infty = 2\|\hat{\mu}\|_\infty.$$

Therefore we get a required contradiction. Q.E.D.

PROOF OF COROLLARY 1. By the well-known fact [7], there exists K a totally disconnected perfect independent set with $\sigma(\neq 0) \in M(K)$ and $\hat{\sigma}(\gamma) \to 0$ as $\gamma \to \infty$. Then K is divided with $K = K_1 \cup K_2$ such that $K_1 \cap K_2 \neq \varnothing$ and K_1, K_2 are nonempty perfect compact subsets with $\tau(\neq 0) \in M(K_1)$ and $\hat{\tau}(\gamma) \mapsto 0$ as $\gamma \mapsto \infty$. Since K is an independent set, $\{K_1 + \{x\}\}_{x \in K_2}$ are pairwise disjoint sets. By [**11**], there exists $K(x)$ a totally disconnected perfect independent and non-Helson set with $K(x) \subset K_1 + \{x\}$. Then $\{K(x)\}_{x \in K_2}$ are an uncountable set, and $\{h_{K(x)}\}_{x \in K_2}$ are what we require. In fact, we assume $x \neq y$ and $K(x) \cap (Gp(K(y)) + z) \neq \varnothing$. Hence, if there exist $w, w' \in K(x)$ such that $w = z + \sum_i n_i k(y, i)$ and $w' = z + \sum_i n_i' k'(y, i)$ where n_i, n_n' integers and $k(y, i)$, $k'(y, i) \in K(y)$, we have $w - w' = -\sum_i n_i' k_i(y, i) + \sum_i n_i k(y, i)$. Since we have $Gp(K(x)) \cap Gp(K(y)) = \{0\}$ by $x \neq y$, we get $w = w'$.

Thus there exists μ a continuous probability measure on $K(x)$ such that $\hat{\mu}(h_{K(y)}) = 0$. Therefore we proved the first half. Also it is well known that there exists K a totally disconnected perfect independent H_1-set in G. So the last half can be proved in the same way as the above method by Theorem 2. Q.E.D.

The proof of Corollary 2 is clear by Theorem 2 and [**13**].

3. *H*-Kronecker set. In this section, we show a generalization of [**12**] in some sense. First we define H-Kronecker set.

DEFINITION. Let G be a nondiscrete LCA group, K a compact set, and H a closed subgroup of G. K is called an H-Kronecker set if, to each $\varepsilon > 0$ and $f \in C(K)$ with $|f| = 1$, there exists $\gamma \in H^\perp$ such that $\|f - \gamma\|_K < \varepsilon$. We simply call K a Kronecker set, if $H = \{0\}$.

Next we shall show the existence of a nontrivial H-Kronecker set.

THEOREM 3. *Let G be a metric I-group, and H a σ-compact closed subgroup with Haar measure zero such that* $\{px: x \in U\} \not\subset H$ *for any natural number p and for any neighborhood U of* $0 \in G$. *Then there exists K an H-Kronecker set which is a totally disconnected perfect set.*

The proof is obtained by modifying [**8**, 5.2].

THEOREM 4. *Under the above assumption, let* ρ *be a nonzero continuous measure on K. Then we have that for any* $\gamma \in \Gamma$, $\mu_i \in M(G)$ $(1 \leqslant i \leqslant n)$, *and* $\varepsilon > 0$, *there exists* $\gamma_0 \in \Gamma$ *such that* $|(h_H\mu_i)\hat{}(\gamma) - (h_H\mu_i)\hat{}(\gamma_0)| < \varepsilon$ $(1 \leqslant i \leqslant n)$ *and* $|\hat{\rho}(\gamma_0)| > \|\rho\|/2$.

PROOF. Let π be the natural mapping from G to G/H, and $\tilde{\pi}$: $M(G) \mapsto M(G/H)$ the mapping induced by π. Then we remark $\tilde{\pi}(\mu)\hat{} = \hat{\mu}|_{H^\perp}$ for all $\mu \in M(G)$. Now by the definition of K, $\pi(K)$ is a Kronecker set on G/H. Hence, by an application of [**12**] in G/H, there exists $\gamma_0 \in H^\perp$ such that

$$\left|\tilde{\pi}(\bar{\gamma}h_H\mu_i)\hat{}(1) - \tilde{\pi}(\bar{\gamma}h_H\mu_i)\hat{}(\gamma_0)\right| < \varepsilon \quad (1 \leqslant i \leqslant n),$$

and $|\tilde{\pi}(\bar{\gamma}\rho)\hat{}(\gamma_0)| > \|\rho\|/2$ (cf. [**4**]).

That is, we have

$$\left|(h_H\mu_i)\hat{}(\gamma) - (h_H\mu_i)\hat{}(\gamma_0 + \gamma)\right| < \varepsilon \quad (1 \leqslant i \leqslant n),$$

and $|\hat{\rho}(\gamma_0 + \gamma)| > \|\rho\|/2$. This establishes the result. Q.E.D.

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