PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 87, Number 1, January 1983

SPACES WHOSE CLOSED IMAGES ARE M_1

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ABSTRACT. Almost local finiteness is introduced. The class of all spaces with a σ-almost locally finite base is an intermediate class between that of free L-spaces and that of M_1-spaces. The class is countably productive, hereditary and the closed image of a space in the class is M_1.

1. Introduction. The M_i-spaces, $i = 1, 2, 3$, were defined in 1961 by J. Ceder [**3**] as natural generalizations of metrizable spaces. C. R. Borges [**1**] later showed that the class of M_3-spaces, which he renamed stratifiable spaces, was a particular well-behaved class of spaces. The equivalence of M_3 and M_2 was proved independently by G. Gruenhage [**4**] and H. Junnila [**8**].

The following is one of the most outstanding problems in general topology: "Does M_3 imply M_1?"

The following are equivalent to this problem by the recent result of R. Heath and H. Junnila [**6**]:

"Is the closed (perfect) image of an M_1-space M_1?"

"Is every (closed) subspace of an M_1-space M_1?"

It is well known that the closed image of an M_3-space is M_3. The perfect irreducible image of an M_1-space was shown to be M_1 by C. R. Borges and D. J. Lutzer [**2**].

On the other hand, free L-spaces were introduced by K. Nagami [**10**] for discussing the dimension theory. A space is a free L-space if and only if it is embedded in the countable product of L-spaces. Every Lašnev space is an L-space and every free L-space is an M_1-space.

In this paper we first introduce the notion of almost local finiteness as a generalization of local finiteness. Then the class of all spaces with a σ-almost locally finite base is investigated. This new class is an intermediate class between that of free L-spaces and that of M_1-spaces. This class contains every space with a σ-closure preserving base of clopen sets which R. Heath and H. Junnila call "M_0-spaces". The class is countably productive, hereditary and, in particular, the closed image of a space in the class is M_1.

All spaces are assumed to be regular T_1. Neighborhoods need not be open. The letter N denotes the set of all natural numbers.

Received by the editors November 4, 1981 and, in revised form, April 1, 1982.

1980 *Mathematics Subject Classification*. Primary 54E20, 54C10; Secondary 54F45.

Key words and phrases. Almost locally finite, σ-almost locally finite base, M_1-space, free L-space, closed map.

0002-9939/82/0000-0557/$02.25

2. Almost local finiteness. In this section we introduce the notion of almost local finiteness as a generalization of local finiteness. The basic properties are studied.

DEFINITION 2.1. Let X be a space, x a point of X, and $\mathcal{A}$ a family of subsets of X. $\mathcal{A}$ is said to be *almost locally finite at* x if there exist a neighborhood U of x and a finite family $\mathcal{B}$ of subsets of X such that $\mathcal{A} | U = \{A \cap U: A \in \mathcal{A}\} \subset \{B \cap V: B \in \mathcal{B}$ and V is a neighborhood of $x\}$. $\mathcal{A}$ is said to be almost locally finite in X if $\mathcal{A}$ is almost locally finite at every point of X.

DEFINITION 2.2. Let X be a space, x a point of X, and $\mathcal{A}$ a family of subsets of X. $\mathcal{A}$ is said to be *closure preserving at* x if for any $\mathcal{A}' \subset \mathcal{A}$, $x \in \mathrm{Cl} \bigcup \{A: A \in \mathcal{A}'\}$ implies $x \in \bigcup \{\mathrm{Cl}\, A: A \in \mathcal{A}'\}$. $\mathcal{A}$ is said to be closure preserving in X if $\mathcal{A}$ is closure preserving at every point of X.

EXAMPLE 2.3. Let X be a space and x a point of X. The following three are typical examples of an almost locally finite family at x:

(1) a family which is locally finite at x;

(2) a family of neighborhoods of x;

(3) a closure preserving open family $\mathcal{A}$ such that for each $A \in \mathcal{A}$, either $x \in A$ or $x \notin \mathrm{Cl}\, A$. (Let $U = X \setminus \bigcup \{\mathrm{Cl}\, A: x \notin \mathrm{Cl}\, A, A \in \mathcal{A}\}$ and $\mathcal{B} = \{X\}$.) In particular, a family of clopen sets which is closure preserving at x.

PROPOSITION 2.4. *Let X be a space, x a point of X and $\mathcal{A}$ an almost locally finite family at x. Then $\mathcal{A}$ is closure preserving at x.*

PROOF. Let $\mathcal{A}' \subset \mathcal{A}$ and $x \in \mathrm{Cl} \bigcup \{A: A \in \mathcal{A}'\}$. By Definition 2.1, there exist a neighborhood U of x and a finite family $\mathcal{B}$ such that $\mathcal{A} | U \subset \{B \cap V: B \in \mathcal{B}$ and V is a neighborhood of $x\}$. Put $\mathcal{B}' = \{B \in \mathcal{B}: B \cap V \in \mathcal{A}' | U$ for some neighborhood V of $x\}$. Then $x \in \mathrm{Cl} \bigcup \{B: B \in \mathcal{B}'\}$. Since $\mathcal{B}'$ is finite, $x \in \mathrm{Cl}\, B_0$ for some $B_0 \in \mathcal{B}'$. By the definition of $\mathcal{B}'$, there exist a member A_0 of $\mathcal{A}'$ and a neighborhood V_0 of x such that $B_0 \cap V_0 = A_0 \cap U$. Then $x \in \mathrm{Cl}(B_0 \cap V_0) = \mathrm{Cl}(A_0 \cap U) \subset \mathrm{Cl}\, A_0$, which completes the proof.

EXAMPLE 2.5. There exists a closure preserving family which is not almost locally finite. Let $X = D \cup \{\infty\}$ be a one point compactification of an infinite discrete space D. Then the family $\mathcal{A} = \{\{d, \infty\}: d \in D\}$ is closure preserving in X but not almost locally finite at the point ∞.

PROPOSITION 2.6. *Let X be a space, $x \in X$ and $\mathcal{A}_\lambda$ an almost locally finite family at x for each $\lambda \in \Lambda$. Let $A_\lambda \supset \bigcup \{A: A \in \mathcal{A}_\lambda\}$. If $\{A_\lambda: \lambda \in \Lambda\}$ is locally finite at x, then the family $\{A: A \in \mathcal{A}_\lambda$ for some $\lambda \in \Lambda\}$ is almost locally finite at x.*

REMARK. The local finiteness of $\{A_\lambda: \lambda \in \Lambda\}$ in this proposition cannot be weakened to almost local finiteness. For example let $X = D \cup \{\infty\}$ be a one point compactification of an infinite discrete space D. For each $d \in D$, let $\mathcal{A}_d = \{\{d\}, X \setminus \{d\}\}$ and $A_d = X$. Then $A_d \supset \bigcup \{A: A \in \mathcal{A}_d\}$ and $\{A_d: d \in D\}$ is almost locally finite in X. But the family $\{A: A \in \mathcal{A}_d$ for some $d \in D\}$ is not closure preserving, hence not almost locally finite at ∞.

The proofs of the following two results are straightforward and are thus omitted.

Proposition 2.7. *Let X and Y be spaces, $x \in X$, $y \in Y$ and let $\mathcal{A}$ be an almost locally finite family at x and $\mathcal{B}$ an almost locally finite family at y. Then the product $\mathcal{A} \times \mathcal{B} = \{A \times B: A \in \mathcal{A}, B \in \mathcal{B}\}$ is almost locally finite at (x, y).*

Proposition 2.8. *Let X be a space, $x \in X$ and $\mathcal{A}$ and $\mathcal{B}$ almost locally finite families at x. Then*:

(1) *for any subset C of X, $\mathcal{A} \mid C$ is almost locally finite at x*;

(2) $\mathcal{A} \wedge \mathcal{B} = \{A \cap B: A \in \mathcal{A}, B \in \mathcal{B}\}$ *is almost locally finite at x*;

(3) *let $\mathcal{A}_\lambda \subset \mathcal{A}$ for each $\lambda \in \Lambda$ and put $A_\lambda = \bigcup \mathcal{A}_\lambda$. Then the family $\{A_\lambda: \lambda \in \Lambda\}$ is almost locally finite at x.*

3. Spaces with a σ-almost locally finite base. Now we are going to study the class of all spaces with a σ-almost locally finite base. Lemma 2.2 of K. Nagami **[11]** suggests this notion to us. We start with theorems asserting that the class is countably productive and hereditary.

Theorem 3.1. *Let X_n be a space with a σ-almost locally finite base for each $n \in N$. Then the product $X = \prod_{n=1}^{\infty} X_n$ has a σ-almost locally finite base.*

Proof. Let $\bigcup\{\mathcal{B}_{n,m}: m \in N\}$ be a σ-almost locally finite base of X_n. For each $k \in N$ and $(n_1, n_2, \ldots, n_k) \in N^k$, put

$$\mathcal{W}(n_1, \ldots, n_k) = \left\{B_1 \times \cdots \times B_k \times \prod_{j=k+1}^{\infty} X_j: B_i \in \mathcal{B}_{i,n_i}, 1 \leqslant i \leqslant k\right\}.$$

Then by Proposition 2.7, each $\mathcal{W}(n_1, \ldots, n_k)$ is almost locally finite in X. Hence $\bigcup\{\mathcal{W}(n_1, \ldots, n_k): (n_1, \ldots, n_k) \in N^k, k \in N\}$ is a σ-almost locally finite base of X.

Theorem 3.2. *Let X be a space with a σ-almost locally finite base and $Y \subset X$. Then the subspace Y has a σ-almost locally finite base.*

Proof. This follows immediately from Proposition 2.8(1).

Theorem 3.3. *The following implications hold: Free L-space $\xrightarrow{(1)}$ Space with a σ-almost locally finite base $\xrightarrow{(2)}$ M_1-space.*

Proof. (2) is obvious from Proposition 2.4.

Let us prove (1). Since every free L-space can be embedded into a countable product of L-spaces, it is enough to show that every L-space has a σ-almost locally finite base. (See **[9]** for the definition of L-spaces.)

Let X be an L-space and $\bigcup\{\mathcal{F}_n: n \in N\}$ a σ-discrete closed network of X. For each $n \in N$, let $\{D(F): F \in \mathcal{F}_n\}$ be a discrete family of open sets such that $F \subset D(F)$. Let $\mathcal{U}_F$ be an approaching anticover of F which is locally finite in $X \setminus F$. Put

$$\mathcal{B}(F) = \big\{B(\mathcal{U}) = F \cup \big(\bigcup\{U: U \in \mathcal{U}\}\big): \mathcal{U} \subset \mathcal{U}_F, \\ B(\mathcal{U}) \subset D(F) \text{ and } B(\mathcal{U}) \text{ is open}\big\}.$$

Then, by Proposition 2.8(3), $\mathfrak{B}(F)$ is almost locally finite at every point of $X \setminus F$. Since every $B \in \mathfrak{B}(F)$ is a neighborhod of F, $\mathfrak{B}(F)$ is almost locally finite at every point of F. Therefore $\mathfrak{B}(F)$ is almost locally finite in X. By Proposition 2.6, $\mathfrak{B}_n = \bigcup\{\mathfrak{B}(F)\colon F \in \mathfrak{F}_n\}$ is almost locally finite in X. Hence $\bigcup\{\mathfrak{B}_n\colon n \in N\}$ is σ-almost locally finite and is obviously a base of X.

THEOREM 3.4. *Let X be a space with a σ-almost locally finite base and F a closed subset of X. Then F has an almost locally finite neighborhood base of open sets.*

PROOF. Let $\bigcup\{\mathfrak{B}_n\colon n \in N\}$ be a σ-almost locally finite base of X. According to Theorem 3.3, X is M_1, which implies that X is perfectly normal. So we can take a family $\{G_n\colon n \in N\}$ of open sets such that $G_1 \supset \operatorname{Cl} G_2 \supset G_2 \supset \operatorname{Cl} G_3 \supset \cdots$, $\bigcap\{G_n\colon n \in N\} = F$. Put $\mathcal{V}_n = \mathfrak{B}_n \mid G_n$. By Proposition 2.8(1), $\mathcal{V}_n$ is almost locally finite. Observe that $\{G_n\colon n \in N\}$ is locally finite at every point of $X \setminus F$. By Proposition 2.6, $\mathcal{V} = \bigcup\{\mathcal{V}_n\colon n \in N\}$ is almost locally finite at every point of $X \setminus F$. Put

$$\mathfrak{W} = \{W(\mathcal{V}') = \bigcup\,\{V\colon V \in \mathcal{V}'\} : \mathcal{V}' \subset \mathcal{V}, F \subset W(\mathcal{V}')\}.$$

Then by Proposition 2.8(3), $\mathfrak{W}$ is almost locally finite at every point of $X \setminus F$. Since every $W \in \mathfrak{W}$ is a neighborhood of F, $\mathfrak{W}$ is almost locally finite at every point of F. Hence $\mathfrak{W}$ is almost locally finite in X. Obviously $\mathfrak{W}$ constitutes a neighborhood base of open sets of F and the proof is completed.

Now we are in a position to show that the closed image of a space with a σ-almost locally finite base is M_1. We need the following theorem of G. Gruenhage.

THEOREM 3.5 [5]. *Suppose X is stratifiable and has the following property*:

(*) *Whenever H and K are closed subsets of X with $H \subset K$, then H has a σ-closure preserving outer base in K.*

Then every closed image of X has the same property and is therefore M_1.

THEOREM 3.6. *Let $f\colon X \to Y$ be a closed onto map and let X have a σ-almost locally finite base. Then Y is an M_1-space.*

PROOF. In the light of Theorem 3.5, it is enough to show that every space X with a σ-almost locally finite base has property (*). Apply Theorems 3.2, 3.4 and Proposition 2.4.

We do not know the answer to the following question:

Question 3.7. Does every perfect (closed) image of a space with a σ-almost locally finite base have a σ-almost locally finite base?

However, by Theorem 3.4, we have

COROLLARY 3.8. *Let X be a space with a σ-almost locally finite base and F a closed subset. Then X/F has a σ-almost locally finite base.*

Now let us ask whether any of the implications of Theorem 3.3 reverse.

EXAMPLE 3.9. There exists a space with a σ-almost locally finite base which is not free L. In [7], M. Itō constructed a free L-space Y containing a closed set E such that

Y/E is not free L. By Corollary 3.8, Y/E has a σ-almost locally finite base. Thus Y/E is the desired example.

The following question remains unanswered:

Question 3.10. Is there an M_1-space which does not have a σ-almost locally finite base?

ACKNOWLEDGEMENTS. We would like to thank the referee for his helpful suggestions.

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