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# ON ALMOST RATIONAL CO-$H$-SPACES

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ABSTRACT. Let $X$ be a 0-connected co-$H$-space whose homotopy groups $\pi_n(X)$ are $Q$ vector spaces if $n > 1$ and whose fundamental group $\pi_1(X)$ is arbitrary. We prove that $X$ is homotopy equivalent to a wedge of rational spheres of dimension at least two and of ordinary one-dimensional spheres.

**Introduction.** We investigate 0-connected but not necessarily 1-connected co-$H$-spaces $X$ with $\pi_n(X)$ a rational vector space for $n \geq 2$. We call such spaces 0-connected *almost rational co-H-spaces* ($\pi_1(X)$ need not be and indeed is not a rational vector space provided it is not trivial). We prove that such a space $X$ is homotopy equivalent to a wedge of rational spheres of dimension bigger than one and of ordinary one-dimensional spheres. Analogous results for simply connected spaces which satisfy certain finiteness conditions were proved by Berstein [**Be**] and Toomer [**To**].

We remark that spaces with $\pi_1(X)$ free and $\pi_n(X)$ a $Q$ vector space for $n \geq 2$ occur quite often, namely as quotients of a rationalization of the universal cover of a space with free fundamental group [**Co**, p. 395f].

The paper is organized as follows: In §1 we prove the result for suspensions of 1-connected rational spaces. Using this we are able to prove the result for $S\Omega X$, if $X$ is a 0-connected almost rational space (§2). If $X$ is a co-$H$-space, it is a retract of $S\Omega X$ and this fact allows a proof of the general result (§3).

**0. Notations and conventions.** (a) All spaces are assumed to be of the homotopy type of a $CW$-complex and all constructions like products etc. are performed in the compactly generated category.

(b) A nilpotent space whose homology groups are rational vector spaces is called a rational space. For basic properties of localization, in particular rationalization, the reader is referred to [**Su** or **Hi-Mi-Ro, 1**].

(c) The rationalization of a sphere $S^n$ ($n > 0$) is called a rational sphere $S^n_Q$. It is obvious that a simply connected space is a rational sphere if and only if it has the homology of a rational sphere.

**1.** PROPOSITION. *Let $X$ be a simply connected rational space. Then $SX$ is up to homotopy a wedge of rational spheres.*

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PROOF. We use a homology decomposition of $X$ (cf. **[Hi]**)

$$X_2 \subset X_3 \subset X_4 \subset \cdots \subset \bigcup_{n \geq 2} X_n = X_\infty.$$

We have

(1) $X_\infty \simeq X$,

(2) all $X_n$ are 1-connected,

(3) $H_r(X_n) = 0$ for $r > n$,

(4) $H_r(X_n) \overset{\cong i_*}{\to} H_r(X_\infty)$ for $r \leq n$ where $i$ denotes the inclusion,

(5) $X_{n+1}$ is up to homotopy obtained from $X_n$ by attaching a Moore space $M(H_{n+1}(X), n)$ by a map $\alpha_n$.

From (5) we obtain a Puppe sequence

$$M(H_{n+1}(X), n) \overset{\alpha_n}{\to} X_n \to X_{n+1} \to M(H_{n+1}(X), n+1) \overset{S\alpha_n}{\to} SX_n \to SX_{n+1}.$$

(Note that $M(H_{n+1}(X), n+1) \simeq SM(H_{n+1}(X), n)$.) Now $M(H_{n+1}(X), n+1)$ is up to homotopy a wedge of rational spheres because $H_{n+1}(X)$ is a rational vector space. We will prove inductively that $SX_{n+1}$ is up to homotopy a wedge of rational spheres by showing that $S\alpha_n \simeq 0$. (Induction starts because $X_2 \simeq M(H_2(X), 2)$.) Then it follows that $SX_\infty$ is up to homotopy a wedge of rational spheres.

It follows from (4) that $\tilde{H}_*(\alpha_n) = 0$. Hence our proposition will follow from

LEMMA. *If $X \overset{f}{\to} Y$ is a map between rational spaces such that $\tilde{H}_*(f) = 0$ then* $Sf \simeq 0$.

PROOF. It suffices to show that $X \overset{f}{\to} Y \overset{i}{\to} \Omega SY$ is null-homotopic where $i$ is the canonical map. Now $\Omega SY$ is a 0-connected rational $H$-space. In such a space all Postnikov invariants are trivial (see **[Mi-Mo**, p. 263]) and thus

$$\Omega SY \simeq \prod_{n=1}^{\infty} K(\pi_n(\Omega SY), n).$$

Here $\prod$ means the direct limit of the finite products. Hence we have only to show that each "factor"

$$f_n \colon X \to Y \to K(\pi_n(\Omega SY), n)$$

is nullhomotopic. Of course, $\tilde{H}_*(f_n) = 0$ and by the universal coefficient theorem we see that

$$\tilde{H}^*(f_n) \colon \tilde{H}^n(K(\pi_n(\Omega SY), n); \pi_n(\Omega SY)) \to \tilde{H}^n(X; \pi_n(\Omega SY))$$

is the zero map. But this is only another way of saying $f_n \simeq 0$.

**2.** LEMMA. *If $X$ is a 0-connected space such that $\pi_n(X)$ is a rational vector space for all $n \geq 2$ then $S\Omega X$ is up to homotopy a wedge of rational spheres of dimension at least two and of ordinary one-dimensional spheres.*

PROOF. (a) Let us first assume that $X$ is 1-connected. Then $\Omega X$ is a 0-connected rational $H$-space and we have

$$\Omega X \simeq \prod_{n \geq 1} K(\pi_n(\Omega X), n).$$

Now we use that $S(Y \times Z) \simeq SY \vee SZ \vee S(Y \wedge Z)$ for any well-pointed spaces $Y$, $Z$. It follows that

$$\begin{aligned} S\Omega X &\simeq S\Big( K(\pi_1(\Omega X), 1) \times \prod_{n \geq 2} K(\pi_n(\Omega X), n)\Big) \\ &\simeq SK(\pi_1(\Omega X), 1) \vee S\Big( \prod_{n \geq 2} K(\pi_n(\Omega X), n)\Big) \\ &\quad \vee S\Big( K(\pi_1(\Omega X), 1) \wedge \prod_{n \geq 2} K(\pi_n(\Omega X), n)\Big). \end{aligned}$$

The Künneth theorem shows that the third summand in the wedge decomposition is a rational space and thus it follows from §1 that the second and third summands are wedges or rational spheres. It remains to look at $SK(\pi_1(\Omega X), 1)$. If $\pi_1(\Omega X)$ is a finite-dimensional $Q$ vector space we use induction on the dimension. Induction starts because $SK(Q, 1) \simeq SS^1_Q \simeq S^2_Q$. Now suppose we know the result if $\pi_1(\Omega X)$ is $n$-dimensional. Then we get, in the case of an $(n+1)$-dimensional $\pi_1(\Omega X)$,

$$\begin{aligned} SK(\pi_1(\Omega X), 1) &\simeq S\Big( \prod_{i=1}^{n+1} K(Q, 1)\Big) \simeq S\Big( \Big( \prod_{i=1}^{n} K(Q, 1)\Big) \times K(Q, 1)\Big) \\ &\simeq S\Big( \prod_{i=1}^{n} K(Q, 1)\Big) \vee SK(Q, 1) \vee S\Big( \Big( \prod_{i=1}^{n} K(Q, 1)\Big) \wedge K(Q, 1)\Big). \end{aligned}$$

The first summand is a wedge of rational spheres by induction hypothesis, the second one is $S^2_Q$ and the third one decomposes as a wedge of rational spheres by §1.

Now we remark that for 1-connected rational spaces $X$ a decomposability as a wedge of rational spheres is equivalent to the Hurewicz-homomorphism $h_X$ being an epimorphism. Using this and passing to the limit shows that $SK(\pi_1(\Omega X), 1)$ decomposes in the case of an arbitrary $\pi_1(\Omega X)$, too.

(b) Now let $X$ be arbitrary. Then we may write

$$\Omega X \simeq (\Omega X)_0 \times \pi_0(\Omega X)$$

where $(\Omega X)_0$ is the path component of the constant path and $\pi_0(\Omega X)$ is equipped with the discrete topology. Then we have

$$S\Omega X \simeq S(\Omega X)_0 \vee S(\pi_0(\Omega X)) \vee S((\Omega X)_0 \wedge \pi_0(\Omega X)).$$

$(\Omega X)_0$ is a connected rational $H$-space, hence a product of Eilenberg-Mac Lane spaces, and the same argument as above shows that $S(\Omega X)_0$ is a wedge of rational spheres. If we write $S((\Omega X)_0 \wedge \pi_0(\Omega X)) \simeq S(\Omega X)_0 \wedge \pi_0(\Omega X)$ we see that this is also a wedge of rational spheres. $S\pi_0(\Omega X)$ is obviously a wedge of ordinary one-dimensional spheres.

3. THEOREM. *If $X$ is a 0-connected almost rational co-$H$-space then $X$ is up to homotopy a wedge of rational spheres of dimension at least two and of ordinary one-dimensional spheres.*

PROOF. *Case* 1. $X$ is 1-connected.

We know from §2 that $S\Omega X$ decomposes in the way desired. Hence $h_{S\Omega X}$ is epic and thus $h_X$ is epic becauses $X$ is a retract of $S\Omega X$ **[Ga]**. This implies that $X$ decomposes.

*Case* 2. $X$ is not 1-connected.

We look at the universal cover $\tilde{X}$. By Lemma 6.2 of **[Hi-Mi-Ro, 2]** we know that $\tilde{H}_*(\tilde{X})$ is a free $Q[\pi_1 X]$ module. Because $X$ is a retract of $S\Omega X$ **[Ga]** it follows that $\tilde{X}$ is a retract of $\widetilde{S\Omega X}$. The latter is up to homotopy a wedge of rational spheres which follows from the decomposability of $S\Omega X$ (§2). Therefore $h_{\widetilde{S\Omega X}}$ is epic and hence $h_{\tilde{X}}$ is as well.

Thus we may choose a $Q[\pi_1 X]$ basis $\{x_\alpha\}$ of $\tilde{H}_*(\tilde{X})$ and represent it by a map $Z := \bigvee_\alpha S_Q^{n(\alpha)} \xrightarrow{f} \tilde{X}$ where $n(\alpha)$ is the dimension of $x_\alpha$.

Let $Y := Z \vee W$ where $W = \bigvee_\beta S^1$ is a wedge of circles such that $\pi_1 W \cong \pi_1 X$. (Note that $\pi_1 X$ is free as a subgroup of the free group $\pi_1(S\Omega X)$.) Let $\tilde{Y}$ be the universal cover of $Y$.

We are going to construct a map $\tilde{g}$: $\tilde{Y} \to \tilde{X}$ which is equivariant with respect to the actions of $\pi_1 Y \cong \pi_1 X$ and is a homotopy equivalence which implies that the induced map $g$: $Y \to X$ is a homotopy equivalence. Roughly speaking, $\tilde{g}$ is chosen as an equivariant extension of $f$: $Z \to \tilde{X}$.

In more detail, $\tilde{Y}$ may be obtained from the universal cover $\tilde{W}$ of $W$ by attaching one copy of $Z$ to each vertex of $\tilde{W}$. We pick one vertex of $\tilde{W}$ which serves as basepoint and take the restriction of $\tilde{g}$ to the corresponding copy of $Z$ to be the map $f$. The requirement of being equivariant defines $\tilde{g}$ on all other copies of $Z$. (Here equivariance means, of course, that we have chosen an isomorphism $\pi_1 Y \cong \pi_1 X$.) So it remains to define $\tilde{g}$ on the edges of $\tilde{W}$. We pick a collection of edges which is in 1-1 correspondence to a basis of $\pi_1 W$ consisting of the one-dimensional spheres in $W$. If such an edge joins the vertices $a$ and $b$ we choose any path in $\tilde{X}$ joining $\tilde{g}(a)$ and $\tilde{g}(b)$. We extend $\tilde{g}$ to the other edges equivariantly. It is now obvious that $\tilde{g}$ is a homology isomorphism and hence induces isomorphisms in homotopy groups. The induced map

$$g\colon Y = \tilde{Y}/\pi_1 Y \to \tilde{X}/\pi_1 X = X$$

induces an isomorphism in $\pi_1$, too, and is thus a homotopy equivalence.

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## REFERENCES

**[Be]** I. Berstein, *Homotopy* mod *C of spaces of category* 2, Comment Math. Helv. **35** (1961), 9–14.

**[Co]** G. Cooke, *Replacing homotopy actions by topological actions*, Trans. Amer. Math. Soc. **237** (1978), 391–406.

**[Ga]** T. Ganea, *Cogroups and suspensions*, Invent. Math. **9** (1970), 185–197.

**[Hi]** P. Hilton, *Homotopy theory and duality*, Gordon & Breach, New York, 1965.

**[Hi-Mi-Ro, 1]** P. Hilton, G. Mislin and J. Roitberg, *Localization of nilpotent groups and spaces*, North-Holland, Amsterdam, 1975.

**[Hi-Mi-Ro, 2]** ______, *On co-H-spaces*, Comment Math. Helv. **53** (1978), 1–14.

**[Mi-Mo]** J. Milnor and J. C. Moore, *On the structure of Hopf algebras*, Ann. of Math. **81** (1965), 211–264.

**[Su]** D. Sullivan, *Geometric Topology*. Part 1: *Localization, periodicity and Galois symmetry*, mimeographed notes, M. I. T., 1970.

**[To]** G. H. Toomer, *Two applications of homology decompositions*, Canad. J. Math. **27** (1975), 323–329.

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