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A CYCLE IS THE FUNDAMENTAL CLASS OF AN EULER SPACE

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ABSTRACT. We prove that every cycle in a closed P.L. manifold M can be regarded as the fundamental class of an Euler subpolyhedron of M.

Let V be a compact real analytic manifold without boundary. It is a long-standing problem to see which ($\mathbf{Z}_2$-) homology classes of V can be represented as the fundamental class of an analytic subset of V (and, in fact, it is conjectured that this is true for any homology class). The analogous problem arises with real algebraic manifolds, although in this case the general statement is false (even if V is connected; see, for instance, [**BT**]).

D. Sullivan (in [**S**]) observed that every real analytic set can be regarded as an Euler space (see definition below); it is then natural to ask, first of all, if it is true that every homology class of a closed P.L. manifold M can be represented as the fundamental class of an Euler subpolyhedron of M.

In this note we prove that this in fact happens: actually, we give a construction to add lower-dimensional simplexes to a cycle in M until we get an Euler space (in M).

The techniques used are entirely elementary and involve merely P.L. transversality (as stated for example in [**RS**]) and combinatorial results on Euler spaces (see [**A**]).

We shall work in the P.L. category. For notations and definitions we refer to [**RS**]. All cycles and manifolds are intended unoriented and compact.

By an *n-cycle P* we mean a polyhedron $P = |K|$ such that

(1) $n = \max\{\dim A, \text{ for } A \text{ a simplex of } K\}$,

(2) each $(n-1)$-simplex of K is the face of an even number of simplexes of K.

By an *n-cycle P with boundary* ∂P we mean a pair of polyhedra $(P, \partial P) = |K, \partial K|$ such that (1) $n = \max\{\dim A, \text{ for } A \text{ a simplex of } K\}$, (2) ∂P is an $(n-1)$-cycle, (3) each $(n-1)$-simplex of $K \setminus \partial K$ is the face of an even number of n-simplexes of K, (4) each $(n-1)$-simplex of ∂K is the face of an odd number of n-simplexes of K. A *cycle* (*with boundary*) *in M* is a subpolyhedron of M which is a cycle (with boundary).

A *closed* (P.L.) manifold is a compact (P.L.) manifold without boundary.

An *Euler space* is a polyhedron P such that, for each $x \in P$, $\chi(\mathrm{lk}(x, P)) \equiv 0 \pmod 2$.

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An *Euler pair* is a pair of polyhedra (P, Q) such that (1) $\forall x \in P \setminus Q$, $\chi(\mathrm{lk}(x, P)) \equiv 0 \pmod 2$; (2) $\forall x \in Q$, $\chi(\mathrm{lk}(x, Q)) \equiv 0 \pmod 2$; (3) $\forall x \in Q$, $\chi(\mathrm{lk}(x, P)) \equiv 1 \pmod 2$.

REMARKS. (1) An Euler space is a cycle (without boundary).

(2) An Euler pair (P, Q) is *not*, in general, a cycle with boundary (if $\dim P = n$, Q may not be of dimension $n - 1$).

(3) Note that the definition of an n-cycle is slightly different from the usual one which requires also each simplex of K to be the face of an n-simplex of K. However, a cycle as we defined it naturally carries a fundamental class (which is a cycle in the usual sense) as follows:

Let $P = |K|$ be an n-cycle. The *fundamental class* $\tilde{P}$ of P is the polyhedron obtained by taking all the n-simplexes of K (together with their faces). Note that, if P is connected, then $\tilde{P} \hookrightarrow P$ is a representative of the generator of $H_n(P; \mathbf{Z}_2) \cong \mathbf{Z}_2$.

In order to show the kind of arguments used, we first prove an "abstract" version of the stated result, that is

THEOREM 1. *Let P be an n-cycle. Then there exists an Euler polyhedron P' such that $P' \supset P$ and* $\dim(P' \setminus P) < n$.

PROOF. Let $P = |K|$ and assume that $K = T^{(1)}$, that is, K is the first barycentric subdivision of another triangulation T of P. Set

$$Q = \overline{\{A \in K : \chi(\mathrm{lk}(A, K)) \equiv 1 \pmod 2\}}\,.$$

$Q = |H|$ is a subpolyhedron of P and $\dim Q < n - 1$ (as P is a cycle).

(a) Assume $\dim Q = 0$. Then Q consists of a finite number of points $v_1, \ldots, v_h$ and (P, Q) is an Euler pair. Let Z be the 1-skeleton of K; then (for the properties of the barycentric subdivision) Z is a 1-cycle with boundary the 0-skeleton of H, that is, Q itself (see [A], Propositions 1 and 2, and the subsequent remark). Thus h is even and we can form $P' = P \cup_Q \Gamma$, where Γ is any 1-cycle with boundary Q.

(b) The general case. Let $d = \dim Q$ $(0 < d \leqslant n - 2)$. We prove first of all that $Q = |H|$ is a d-cycle. Let A be a $(d-1)$-simplex of H and $B_1, \ldots, B_h$ the set of d-simplexes of H such that $B_i > A$. If C is a simplex of $R = \mathrm{lk}(A, K)$, then $C * A \in K$ and $\mathrm{lk}(C, R) = \mathrm{lk}(C * A, K)$ (here $*$ denotes the join operation). Since $\dim(C * A) = \dim C + d$, $\chi(\mathrm{lk}(C, R))$ is always even, except for the vertices $v_1, \ldots, v_h$ such that $v_i * A = B_i$. Then, by the case (a), h is even, which means that Q is a cycle.

Now we can form $P_1 = P \cup_Q \Gamma$, where Γ is any $(d+1)$-cycle with boundary Q, for example the cone on Q. P_1 is not necessarily an Euler space; however, if B is a d-simplex of H, $\mathrm{lk}(B, P_1) = \mathrm{lk}(B, P) \amalg \{\text{odd number of points}\}$, so that $Q_1 = \overline{\{A \in P_1 : \chi(\mathrm{lk}(A, P_1)) \equiv 1\}}$ is a subpolyhedron of dimension $\leqslant (d-1)$ in P_1; by iterating the argument we obtain the required Euler space P'. $\square$

Note that the hypothesis that P is a cycle is necessary; see, for example, the following Figure 1.

The difficulty which arises in the general case is essentially to prove that Q is now a boundary in the ambient manifold.

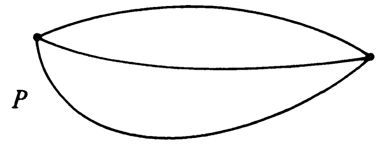

FIGURE 1

THEOREM 2. *Let M be a closed m-manifold and P a cycle of dimension $n < m$ in M. Then there exists a subpolyhedron P', $P \subset P' \subset M$, such that P' is an Euler space and* $\dim(P' \setminus P) < n$.

PROOF. Let Q be defined as in the previous theorem and (L, K, H) be a triangulation of (M, P, Q) which we assume, for the sake of simplicity, to be the first barycentric subdivision of another triangulation of (M, P, Q) (see remark below).

CLAIM. Q is a boundary in P.

(Note that this has already been proved in the case $\dim Q = 0$.) Let $d = \dim Q$; let N be the simplicial neighbourhood of $H^{(1)}$ in $K^{(1)}$, $\dot{N}$ the boundary of N, p: $N \to Q$ the simplicial retraction and $\dot{p} = p \mid \dot{N}$. $(\overline{P \setminus N}, \dot{N})$ is an Euler pair; therefore (again by [A, Proposition 1]), if Z denotes the $(d+1)$-skeleton of $\overline{P \setminus N}$ and S denotes the d-skeleton of $\dot{N}$ (both with respect to $K^{(1)}$), we have that Z is a $(d+1)$-cycle with boundary S. Let $f = \dot{p} \mid S$; f is a simplicial map and we want to show that its degree is odd. Let $\sigma \in H^{(1)}$ be a d-simplex and $A \in H$ such that $\sigma \subset A$; we must prove that $\#\{\text{simplexes in } f^{-1}(\mathring{\sigma})\} = \#\{d\text{-simplexes in } \dot{p}^{-1}(\mathring{\sigma})\}$ is odd; as

$$\#\{B \in K: A < B\} = \#\{\text{simplexes } C \text{ of } \operatorname{lk}(A, K)\}$$
$$\equiv \chi(\operatorname{lk}(A, K)) \equiv 1 \pmod 2,$$

it is enough to show that, for each $B > A$, $\#\{d\text{-simplexes in } \dot{p}^{-1}(\mathring{\sigma}) \cap \mathring{B}\}$ is odd. Let $B > A$; then $B = A * C$ and $\dot{p} \mid \dot{N} \cap B$: $\dot{N} \cap B \to A$ is obtained by the pseudoradial projection from C.

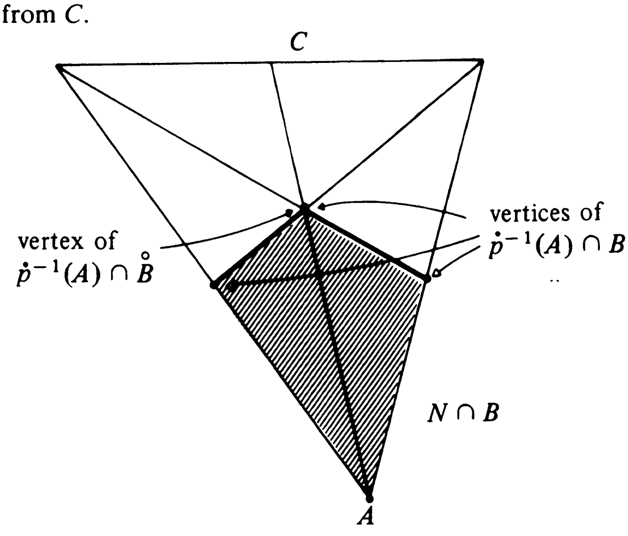

FIGURE 2

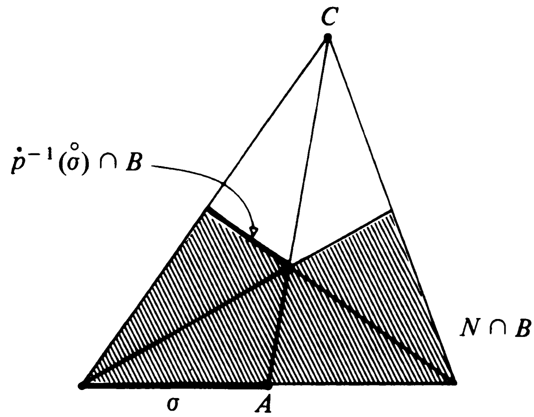

FIGURE 3

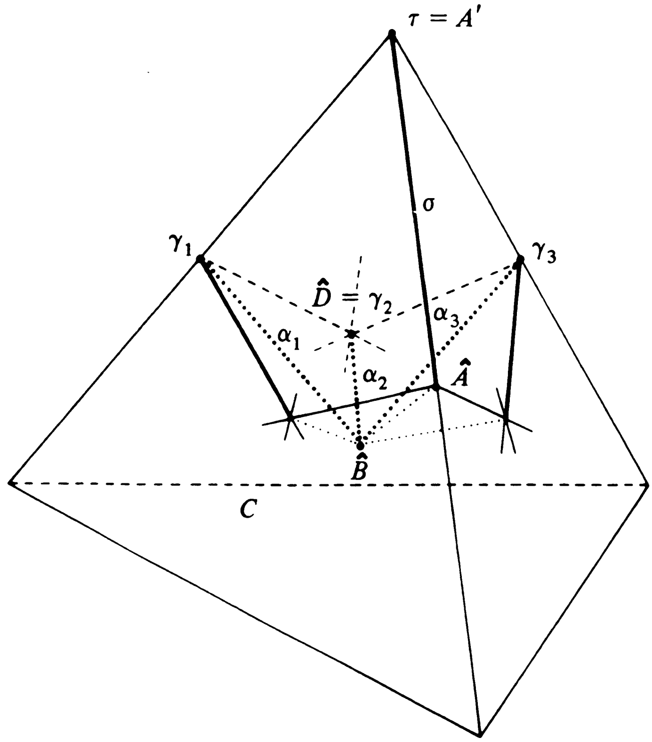

FIGURE 4

Note that, if $\dim A = 0$, $\#\{\text{vertices of } \dot{p}^{-1}(A) \cap \mathring{B}\} = 1$ and $\#\{\text{vertices of } \dot{p}^{-1}(A) \cap B\} = \#\{\text{vertices of } C \text{ in } K^{(1)}\} \equiv 1 \pmod 2$ (see Figure 2); while, if $\dim C = 0$ (so that B is a cone over A with vertex C), $\#\{d\text{-simplexes in } \dot{p}^{-1}(\mathring{\sigma}) \cap \mathring{B}\} = \#\{d\text{-simplexes in } \dot{p}^{-1}(\mathring{\sigma}) \cap B\} = 1$ (see Figure 3). In general, if $\sigma = \hat{A} * \tau$, let A'

be the face of A containing τ and $D = C * A'$. Then, if α is a d-simplex in $\dot{p}^{-1}(\mathring{\sigma}) \cap \mathring{B}$, necessarily $\alpha = \hat{B} * \gamma$, where γ is a $(d-1)$-simplex in $\dot{p}^{-1}(\tau) \cap D$ (see Figure 4). In order to conclude by induction, we have to show that also $\#\{d$-simplexes in $\dot{p}^{-1}(\mathring{\sigma}) \cap B\}$ is odd. But, if C' varies over the faces of C, and $B' = A * C'$, then

$$\#\{d\text{-simplexes in } \dot{p}^{-1}(\mathring{\sigma}) \cap B\} = \#\{d\text{-simplexes in } \dot{p}^{-1}(\mathring{\sigma}) \cap \mathring{B}\} + \sum_{C' < C} \#\{d\text{-simplexes in } p^{-1}(\mathring{\sigma}) \cap \mathring{B}'\},$$

By induction, all the terms of this sum are odd; moreover, their number equals $\#\{C'\colon C' \leqslant C\} \equiv 1 \pmod 2$. Thus $f\colon S \to Q$ is an odd degree map, so that the mapping cylinder C_f is a $(d+1)$-cycle in P with boundary $S \amalg Q$ and $Q' = Z \cup_S C_f$ is the required cycle with boundary Q. This proves the claim.

In order to prove the theorem, it is enough now to put Q' transverse to P in M relatively to Q (see [**RS**, Theorem 5.3]). In this way we get a cycle Q'' in M with boundary Q and such that $\dim(Q'' \cap P) \leqslant d + 1 + n - m \leqslant d$. Form $P_1 = P \cup_Q Q''$; P_1 is an n-cycle in M and, if A is a d-simplex in P_1, then

$$\operatorname{lk}(A, P_1) = \begin{cases} \operatorname{lk}(A, P) \amalg \{\text{odd number of points}\} & \text{if } A \in Q, \\ \operatorname{lk}(A, P) & \text{if } A \in P \setminus Q'', \\ \operatorname{lk}(A, Q'') & \text{if } A \in Q'' \setminus P, \\ \operatorname{lk}(A, P) \amalg \{\text{even number of points}\} & \text{if } A \in Q'' \cap P. \end{cases}$$

In each case $\chi(\operatorname{lk}(A, P_1)) \equiv 0$, so that $Q_1 = \overline{\{A \in P_1\colon \chi(\operatorname{lk}(A, P_1)) \equiv 1\}}$ has dimension $\leqslant (d-1)$ and we can iterate the argument as before until we get an Euler space P'. □

Remark. As regards the choice of the triangulation, what we need is only that the simplicial neighbourhood N of Q in P (with respect to $K^{(1)}$) is in fact a regular neighbourhood; therefore, any triangulation (K, H) such that Q is full in P would be enough (see [**RS**] for a definition of full).

Corollary. *Every homology class $z \in H_n(M, \mathbf{Z}_2)$ can be represented as the fundamental class of an Euler subpolyhedron of dimension n in M.*

Addendum. With respect to the problem stated in the introduction (that is, to represent $\mathbf{Z}_2$-homology classes of a real algebraic manifold by algebraic subvarieties), since this paper was written we have proved the following (see [**BD**]):

For each $d \geqslant 11$, there exists a compact smooth manifold V and a class $z \in H_{d-2}(V, \mathbf{Z}_2)$ such that, for any homeomorphism $h\colon V \to V'$ between V and a real algebraic manifold V', $h_*(z) \in H_{d-2}(V', \mathbf{Z}_2)$ cannot be represented by an algebraic subvariety of V'.

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