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# GEOMETRIC REALIZATION OF A FINITE SUBGROUP OF $\pi_0 \varepsilon(M)$. II

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ABSTRACT. Let $M$ be a closed aspherical manifold with a virtually 2-step nilpotent fundamental group. Then any finite group $G$ of homotopy classes of self-homotopy equivalences of $M$ can be realized as an effective group of self-homeomorphisms of $M$ if and only if there exists a group extension $E$ of $\pi$ by $G$ realizing $G \to \operatorname{Out} \pi_1 M$ so that $C_E(\pi)$, the centralizer of $\pi$ in $E$, is torsion-free. If this is the case, the action $(G, M)$ is equivalent to an affine action $(G, M')$ on a complete affinely flat manifold homeomorphic to $M$. This generalizes the same result for flat Riemannian manifolds.

Let $M$ be a closed aspherical manifold, $\varepsilon(M)$ the group of homotopy equivalences of $M$ into itself. Any $f \in \varepsilon(M)$ induces an isomorphism $f_*\colon \pi_1(M, x) \to \pi_1(M, f(x))$. By choosing a path $\omega$ from $x$ to $f(x)$, we get an automorphism $f_*^\omega$ of $\pi_1(M, x)$, mapping $[\tau]$ to $[\omega^{-1}(f \circ \tau)\omega]$ for any loop $\tau$ based at $x$. A different choice of $\tau$ alters $f_*^\omega$ by an inner automorphism of $\pi_1(M, x)$. Therefore, we obtain a natural surjective homomorphism $\Psi\colon \varepsilon(M) \to \operatorname{Out} \pi_1 M$, where $\operatorname{Out} \pi_1 M = \operatorname{Aut} \pi_1 M / \operatorname{Inn} \pi_1 M$, the group of automorphism classes. Of course, $\Psi(f) = [f_*^\omega]$ for any $\omega$ as above. Since we are considering only $\pi_1(M, x)$, we need not specify the base point, so we write $\pi_1 M$ instead of $\pi_1(M, x)$. It is well known that for a closed aspherical manifold $M$, the kernel of $\Psi$ is $\varepsilon_0(M)$, the group of self-homotopy equivalences of $M$ homotopic to the identity. This implies that $\Psi$ factors through $\pi_0 \varepsilon(M)$ so that

$$\begin{array}{ccccc} & & \varepsilon(M) & & \\ & \swarrow & & \searrow \Psi & \\ \pi_0 \varepsilon(M) & & \underset{\cong}{\overset{\Psi_*}{\to}} & & \operatorname{Out}(\pi_1 M) \end{array}$$

commutes. A pair of groups $G$, $\pi$ together with a homomorphism $\phi\colon G \to \operatorname{Out} \pi$ is called an *abstract kernel*, and is denoted by $(G, \pi, \phi)$. Therefore, an injective abstract kernel $(G, \pi_1 M, \phi)$ for a closed aspherical manifold $M$ is really a group of homotopy classes of self-homotopy equivalences of $M$.

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Restricting $\Psi$ to $H(M)$, the group of self-homeomorphisms of $M$, we have a homomorphism $\Psi\colon H(M) \to \operatorname{Out} \pi_1 M$. This is surjective since $M$ is a closed aspherical manifold. A *geometric realization* of an abstract kernel $\phi\colon G \to \operatorname{Out} \pi_1 M$ is a homomorphism $\hat{\phi}\colon G \to H(M)$ so that $\Psi \circ \hat{\phi} = \phi$. In this paper we will require $\hat{\phi}$ to be injective so that the action of $G$ on $M$ is effective. Therefore, when the abstract kernel is injective, the problem reduces to the problem of geometric realization of a group of homotopy classes of self-homotopy equivalences as a group of self-homeomorphisms.

Suppose we have a $G$-action. Let $\mathfrak{L}(G, M)$ be the group of all liftings of elements of $G$ to $\tilde{M}$, the universal covering of $M$. Then

$$1 \to \pi_1 M \to \mathfrak{L}(G, M) \to G \to 1$$

is exact and admissible (i.e., $C_{\mathfrak{L}(G,M)}(\pi_1 M)$, the centralizer of $\pi_1 M$ in $\mathfrak{L}(G, M)$, is torsion-free). See [**LR1**, Lemma 1]. Therefore, in order that an abstract kernel $(G, \pi_1 M, \phi)$ for a closed aspherical manifold $M$ be realized geometrically, it is necessary that the abstract kernel have an extension which is admissible. In certain cases, this condition is also sufficient when $G$ is finite. For example, it is true for closed flat Riemannian manifolds (see [**LR1**; **ZZ**] when $\phi$ is injective), for certain Seifert fibered spaces [**R**], and for Seifert relatives of compact flat Riemannian manifolds [**L2**]. In this paper we will verify that this is also the case for closed aspherical manifolds with virtually 2-step nilpotent fundamental groups. In a subsequent joint paper with Frank Raymond we shall generalize this to more general Seifert fibered spaces.

In [**L3**] the following is proved.

FACT 1 [**L3**]. A closed aspherical manifold $M$ of dimension $\neq 3$ or 4 with a virtually 2-step nilpotent fundamental group is homeomorphic to a complete affinely flat manifold $M'$.

We now state our main theorem.

THEOREM. *Let $M^m$ be a closed aspherical manifold $(m \neq 3, 4)$ with a virtually 2-step nilpotent fundamental group. A finite abstract kernel $\phi\colon G \to \operatorname{Out} \pi_1 M$ can be realized as an effective group of self-homeomorphisms of M if and only if it admits an admissible extension. In fact, $(G, M)$ is equivalent to an affine action $(G, M')$ on a closed affinely flat manifold $M'$ homeomorphic to M.*

PROOF OF THE THEOREM. We will need the following facts.

FACT 2 [**FH**]. Let $N^m$ $(m \neq 3, 4)$ be a closed connected infranilmanifold and $M^n$ be an aspherical manifold with $\pi_1(M^n)$ isomorphic to $\pi_1(N^m)$. Then $M^m$ and $N^m$ are homeomorphic.

FACT 3 [**L2**]. Let $Q$ be a virtually free abelian group (of rank $n$) which has a representation $\phi \times \rho\colon Q \to \mathrm{GL}(k, \mathbf{Z}) \times E(n)$ such that image($\phi$) is finite, image($\rho$) is a crystallographic group (of rank $n$). Then for any extension $E$ of $\mathbf{Z}^k$ by $Q$

realizing $\phi$, there exists a homomorphism $f\colon E \to L \circ Q_1 \subset A(k+n)$ so that

$$\begin{array}{ccccccccc} 1 & \to & Z^k & \to & E & \stackrel{p}{\to} & Q & \to & 1 \\ & & \downarrow & & \downarrow f & & \downarrow \phi \times \rho & & \\ 1 & \to & L & \to & L \circ Q_1 & \to & Q_1 & \to & 1 \end{array}$$

commutes. Moreover, $f$ is injective if and only if $E_K = \text{kernel}((\phi \times \rho) \circ p)$ is torsion-free.

In the above, $E(n)$ denotes the group of rigid motions on $\mathbf{R}^n$, $A(k+n)$ the group of affine motions on $\mathbf{R}^{k+n}$, $L = L(\mathbf{R}^n, \mathbf{R}^k)$ is the group of all affine maps of $\mathbf{R}^n$ into $\mathbf{R}^k$ and $Q_1 = \text{image}(\phi \times \rho)$. A discrete subgroup $P$ of $E(n)$ with $\mathbf{R}^n/P$ compact is called a crystallographic group of rank $n$.

Suppose the abstract kernel $\phi\colon G \to \text{Out}\,\pi$ $(\pi = \pi_1 M)$ admits an admissible extension $E$ so that $1 \to \pi \to E \to G \to 1$ is exact and $C_E(\pi)$ is torsion-free. Let $N$ be the nilradical of $\pi$, that is, $N$ is the maximal nilpotent normal subgroup of $\pi$. Then $N$ is characteristic in $\pi$. Since $E$ is also virtually 2-step nilpotent, $N$ is an extension of a torsion-free nilpotent subgroup by a finite group. In addition, $N$ is torsion-free. This implies that $N$ itself is 2-step nilpotent. The center $z(N)$ of $N$ is normal in $E$. Since $E/z(N)$ contains a free abelian subgroup $N/z(N)$, it is virtually free abelian. Certainly the natural homomorphism $\phi\colon E/z(N) \to \text{Aut}(z(N)) = \text{GL}(k, \mathbf{Z})$ ($k =$ rank of $z(N)$) has a finite image. There exists a homomorphism $\rho\colon E/z(N) \to E(n)$ $(n = m - k)$ with crystallographic group as image. See [**LR1**] for the proof. Clearly kernel$(\rho)$ is finite. Let $E_K$ be the preimage of kernel$(\phi \times \rho)$ under the natural homomorphism $E \to E/z(N)$. We shall prove that $E_K$ is torsion-free.

First we show $E_K \subset C_E(N)$. Suppose $x \in E_K$. Then $[x, N] \subset z(N)$, $[x, z(N)] = 1$ and $x^t \in z(N)$ for some $t > 0$. Here, $[x, y] = xyx^{-1}y^{-1}$. Let $X$ be the subgroup of $E$ generated by $x$. The first two conditions imply that $X \cdot N$ is again 2-step nilpotent. Since $x^t \in z(N)$, $[x^t, N] = 1$. $[x^t, \alpha] = [x, \alpha]^t$ for all $\alpha \in N$ since $X \cdot N$ is 2-step nilpotent. Note that $[x, \alpha] \in N \subset \pi$. Therefore $[x, \alpha]^t = [x^t, \alpha] = 1$ with $[x, \alpha] \in \pi$. Since $\pi$ is torsion free, $[x, \alpha] = 1$. That is, $x \in C_E(N)$.

Next we show that $\text{Tor}(C_E(N)) = \text{Tor}(C_E(\pi))$. Suppose $\theta$ is an automorphism of $\pi$ such that $\theta|_N$ is the identity. We claim that $\theta(\sigma) \cdot \sigma^{-1} \in C_E(N)$ for all $\sigma \in \pi$. For any $\alpha \in N$,

$$\sigma \cdot \alpha \cdot \sigma^{-1} = \theta(\sigma \cdot \alpha \cdot \sigma^{-1}) = \theta(\sigma) \cdot \theta(\alpha) \cdot \theta(\sigma)^{-1} = \theta(\sigma) \cdot \alpha \cdot \theta(\sigma)^{-1}.$$

This implies that $\theta(\sigma) \cdot \sigma^{-1} \in C_\pi(N)$, since $C_\pi(N)$ is characteristic in $\pi$. Now let $c \in C_E(N)$ be an element of finite order, say, $c^r = 1$ for some $r > 0$. We apply the results above to the automorphism $\theta =$ conjugation by $c$. Certainly $\theta$ induces the identity on $N$, for $c \in C_E(N)$. Therefore $[c, \sigma] = \theta(\sigma) \cdot \sigma^{-1} \in C_\pi(N)$. We claim that $C_\pi(N) = z(N)$. Since $C_\pi(N)$ is a torison-free central extension of $z(N)$ by a finite group, it is again free abelian of rank $k = \text{rank}\, z(N)$ (see [**LR1**, Fact 2]). Therefore, $C_\pi(N) \cdot N$ is a nilpotent normal subgroup of $\pi$ containing $N$. However, $N$ is the nilradical of $\pi$, which implies $C_\pi(N) \subset N$. For any $\sigma \in \pi$,

$$[c^r, \sigma] = [c, \sigma]^{c^{r-1}} \cdots [c, \sigma]^{c^2} \cdot [c, \sigma]^{c} \cdot [c, \sigma]$$

where $x^y = yxy^{-1}$. Since $[x, \sigma] \in N$ and $c \in C_E(N)$, $c[c, \sigma]c^{-1} = [c, \sigma]$. Therefore, $1 = [c^r, \sigma] = [c, \sigma]^r$. As $N$ is torsion-free $[c, \sigma] = 1$, i.e. $c \in C_E(\pi)$. Consequently $\mathrm{Tor}(E_K) \subset \mathrm{Tor}(C_E(N)) = \mathrm{Tor}(C_E(\pi))$ and hence, $E_K$ is torsion-free (since so is $C_E(\pi)$).

Thus we have $\phi$: $E/z(N) \to \mathrm{GL}(k, Z)$ with a finite image, $\rho$: $E/z(N) \to E(n)$ with a finite kernel and with a crystallographic group (of rank $n$) as image. Furthermore, we have just proved that $E_K =$ preimage of kernel$(\phi \times \rho)$ under $E \to E/z(N)$ is torsion-free. Applying Fact 3, we obtain an embedding $f$ of $E$ into $A(m)$. Via $f$, $\pi$ acts on $\mathbf{R}^m$ as a covering transformation yielding $M' = \mathbf{R}^m/f(\pi)$, a compact complete affinely flat manifold. Furthermore, $G = E/\pi$ acts on $M'$ as a group of affine diffeomorphisms. By Fact 2, $M$ and $M'$ are homeomorphic. Then the $G$-action on $M'$ can be pulled back to one on $M$ via this homeomorphism. Q.E.D.

EXAMPLE. A finite subgroup $G$ of Out $\pi_1 M$ cannot be realized as a group of affine actions on $M$ in general. Let $\pi = \langle t_1, t_2, t_3' \rangle$ be the subgroup of $A(3)$, $t_1 = (I, e_1)$, $t_2 = (I, e_2)$ and $t_3' = (T, e_3)$ with $T = I + E_{21}$. Consider the subgroup $\mathbf{Z}_2$ of Out $\pi$ generated by the automorphism $t_1 \to t_1^{-1}$, $t_2 \to t_3'$, $t_3' \to t_2$. This $\mathbf{Z}_2 \subset$ Out $\pi$ can be realized as a group of homeomorphisms but not as a group of affine maps. For if the latter were true, then the derivatives of $t_2$ and $t_3'$ would be similar $3 \times 3$ matrices. But by construction these derivations are $I$ and $T$ respectively, and these matrices are definitely not similar.

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