

## ALGEBRAIC KNOTS ARE ALGEBRAICALLY DEPENDENT

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**ABSTRACT.** Algebraic knots are linearly dependent in  $G_-$ , the algebraic knot concordance group. An example of a linear relation between four algebraic knots is constructed.

An algebraic knot is any one component link of an isolated singularity of a complex curve. Such knots have been classified [4]. Rudolph [3] asked for a description of the subgroup of the knot concordance group generated by this class of knots, and whether or not these knots form an independent set. Litherland [2] showed that signature functions are not sufficient to rule out linear relations among them. In this note we will produce an example showing that no algebraic concordance invariants will suffice. We will prove:

**PROPOSITION.** *The algebraic knots are linearly dependent in  $G_-$ , the algebraic knot concordance group.*

**1. Notation.** A *satellite knot* is any knot in  $S^3$  whose complement contains an essential torus. Fix an oriented satellite knot  $S$  and an essential torus  $T$  in the complement of  $S$ . Let  $V$  denote the solid torus bounded by  $T$ . Note that  $V$  contains  $S$ . The core of  $V$ , called the *companion* of  $S$  (associated with  $T$ ) will be denoted by  $C$ . Define the *winding number*  $w$  of  $S$  of the homology relation  $S \sim wC$  in  $V$ . Orient  $C$  so that  $w \geq 0$  (there is a choice to be made when  $w = 0$ ). Finally, set  $E = f(S)$  where  $f: V \rightarrow S^3$  is an orientation and longitude preserving embedding onto an unknotted solid torus in  $S^3$ . We shall call  $E$  the *embellishment* of  $S$ .

Set  $\Lambda = \mathbb{Z}[t, t^{-1}]$  and  $\Lambda_0 = \mathbb{Q}(t)$ , the quotient field of  $\Lambda$ . For any oriented knot  $K$ , write  $A_K$  for the Alexander module of  $K$ , and  $B_K$  for the Blanchfield pairing on  $A_K$  (= linking pairing  $A_K \times A_K \rightarrow \Lambda_0/\Lambda$ ).  $A_K$  has a square presentation matrix with entries in  $\Lambda$ . Any such matrix  $A_K(t)$  is called the *Alexander matrix* of  $K$ . The associated matrix  $B_K(t)$  for  $B_K$  (with entries in  $\Lambda_0/\Lambda$ ) is called the associated *Blanchfield matrix*.

**2. Example dependence relation.** The following result is implicit in Kearton [1].

**THEOREM.** *Let  $S$  be a satellite knot with core  $C$ , embellishment  $E$  and winding number  $w$ . Then*

$$B_S(t) = B_E(t) \oplus B_C(t^w).$$

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EXAMPLE. Let  $(p_1, q_1; p_2, q_2)$  denote the  $(p_1, q_1)$  cable about the  $(p_2, q_2)$  torus knot. (The winding number of this satellite knot is  $q_1$  with respect to  $(p_2, q_2)$ .) Let  $J = (13, 2; 3, 2) \# (15, 2)$  and  $K = (15, 2; 3, 2) \# (13, 2)$ . According to [4] these are both connected sums of algebraic knots. (The  $(p, q)$  torus knot is always algebraic, and  $(p_1, q_1; p_2, q_2)$  is algebraic if  $p_1 > q_1 p_2 q_2$ .) It follows from the above theorem that

$$B_J(t) = B_{(13,2)}(t) + B_{(3,2)}(t^2) + B_{(15,2)}(t) = B_K(t).$$

According to [5], as  $J$  and  $K$  have the same Blanchfield pairing they are  $S$ -equivalent, and therefore algebraically concordant. Therefore, the four prime factors of  $J$  and  $K$  (which are distinct) satisfy a relation in  $G_-$ .

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