ROUGH AND STRONGLY ROUGH NORMS ON BANACH SPACES

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ABSTRACT. We give equivalent conditions for a norm on a Banach space X to be rough using the sets $A(x)$ defined for each $x \in X$ by $A(x) = \{f \in X^* \mid f(x) = \|x\|,\ \|f\| = 1\}$. This enables us to obtain unitary characterizations for rough and strongly rough norms.

Let X be a real Banach space, and for $x, y \in X$, let $\tau(x, y)$ denote the one sided Gateaux differential of $\|\cdot\|$ at x in the direction y, i.e.,

$$\tau(x, y) = \lim_{t \to 0^+} t^{-1}(\|x + ty\| - \|x\|).$$

For $x_0 \in X$ and $r > 0$ we denote $B_X(x_0, r) = \{x \in X \mid \|x - x_0\| \leqslant r\}$, $B_X = B_X(0, 1)$ and $S_X = \{x \in X \mid \|x\| = 1\}$. For each $x \in X$ let us denote $A(x) = A_X(x) = \{f \in S_{X^*} \mid f(x) = \|x\|\}$, where X^* is the dual space of X. For $f \in S_{X^*}$, let $A^{-1}(f) = \{x \in S_X \mid f(x) = 1\}$ and denote by $\mathrm{Dom}(A^{-1})$ the set $\{f \in S_{X^*} \mid A^{-1}(f) \neq \varnothing\}$.

By the Ascoli-Mazur Theorem [4, p. 447], for $x, y \in X$ we have

$$(1) \qquad\qquad \tau(x, y) = \max\{f(y) \mid f \in A(x)\}.$$

DEFINITION 1 [8, 9, p. 120]. A norm of a Banach space X is said to be *rough* if there is an $\varepsilon > 0$ such that for every $x \in X$ and $\delta > 0$, there exist $x_1, x_2, u \in X$, $\|x_i - x\| < \delta$, $i = 1, 2$, $u \in S_X$, with $\tau(x_2, u) - \tau(x_1, u) \geqslant \varepsilon$.

There are several known characterizations of a Banach space with a rough norm (see e.g., [1, 7, 9, 11, 13]). We add to these characterizations some new ones. In what follows we shall use the following results of [7, Proposition 1, (i) ⇔ (iv), Proposition 1′, (ii) ⇔ (iv)], some of them being implicitly contained in [9, p. 122].

PROPOSITION 1 [7, p. 336]. *The following properties of a given norm of X are equivalent*:

(i) *The norm of X (the dual norm of X^*) is not rough.*

(ii) *For every $\varepsilon > 0$ there is an $x \in S_X$ ($f \in S_{X^*}$) such that whenever $f_n, g_n \in S_X$, $\lim f_n(x) = \lim g_n(x) = 1$ ($x_n, y_n \in S_X$, $\lim f(x_n) = \lim f(y_n) = 1$), then*

$$\limsup\|f_n - g_n\| \leqslant \varepsilon \qquad (\limsup\|x_n - y_n\| \leqslant \varepsilon).$$

For a bounded set $A \subset X$, we denote the diameter of A by $\mathrm{diam}\, A = \mathrm{diam}_X A = \sup\{\|a_1 - a_2\| \mid a_1, a_2 \in A\}$.

THEOREM 1. *The following properties of a given norm of X are equivalent*:

(i) $\|\cdot\|$ *is rough.*

(ii) *There is an $\varepsilon > 0$ such that for every $x \in S_X$ and $\delta > 0$, there exists $y \in S_X$, $\|y - x\| < \delta$ with $\mathrm{diam}(A(x) \cup A(y)) \geqslant \varepsilon$.*

PROOF. (i) $\Rightarrow$ (ii). Since $\|\cdot\|$ is rough, choose $\varepsilon > 0$ given by Definition 1, and let $\varepsilon' = \varepsilon/2$, $x \in S_X$ and $0 < \delta < 2$. By (i) for this $x \in S_X$ and $\delta/2$, there exist $x_1, x_2, u \in X$, $\|x_i - x\| < \delta/2$, $i = 1, 2$, $u \in S_X$, with $\tau(x_2, u) - \tau(x_1, u) \geqslant \varepsilon$. By (1) there exist $f_i \in A(x_i)$, $i = 1, 2$, such that $\tau(x_i, u) = f_i(u)$, $i = 1, 2$. Then

$$\varepsilon \leqslant \tau(x_2, u) - \tau(x_1, u) = f_2(u) - f_1(u) \leqslant \|f_1 - f_2\|.$$

Since $x_i \neq 0$, let $y_i = x_i/\|x_i\|$, $i = 1, 2$. Let $f \in A(x)$. Then $2\varepsilon' \leqslant \|f_1 - f_2\| \leqslant \|f_1 - f\| + \|f - f_2\|$ and so we can suppose (interchanging the indices if necessary) that $\varepsilon' \leqslant \|f_1 - f\| \leqslant \mathrm{diam}(A(x) \cup A(y_1))$, i.e., we have (ii).

(ii) $\Rightarrow$ (i). Let $\varepsilon > 0$ be given by (ii) and let $\varepsilon' = \varepsilon/2$ and $x \in S_X$. By (ii), for each n there exists $y_n \in S_X$, $\|y_n - x\| < 1/n$ with $\mathrm{diam}(A(x) \cup A(y_n)) \geqslant \varepsilon$. Let $f_n, g_n \in A(x) \cup A(y_n)$ such that $\|f_n - g_n\| \geqslant 3\varepsilon/4$ for each n. Clearly, $\lim f_n(x) = \lim g_n(x) = 1$ and $\limsup \|f_n - g_n\| > \varepsilon'$, whence by Proposition 1, the norm is rough.

REMARK 1. Using Theorem 1, it is easy to show that the norm of X is rough if and only if there is an $\varepsilon > 0$ such that for every $x \in S_X$, $f \in A(x)$ and $\delta > 0$, there exists $y \in S_X$, $\|y - x\| < \delta$ with $\sup_{g \in A(y)} \|f - g\| \geqslant \varepsilon$.

COROLLARY 1. *The following properties of a given norm of X are equivalent*:

(i) $\|\cdot\|$ *is rough.*

(ii) *There is an $\varepsilon > 0$ such that for every $x \in S_X$ and $\delta > 0$, there exist $x_1, x_2, u \in S_X$, $\|x_i - x\| < \delta$, $i = 1, 2$, such that $\tau(x_2, u) + \tau(x_1, -u) \geqslant \varepsilon$.*

(iii) *The same as (ii) but with $x_1 = x_2$ or one of x_1, x_2 equals x.*

PROOF. (i) $\Rightarrow$ (ii). This follows by Definition 1, using the well-known fact [4] that $-\tau(x_1, u) \leqslant \tau(x_1, -u)$.

(ii) $\Rightarrow$ (i). Using formula (1) we obtain that (ii) implies condition (ii) of Theorem 1, whence the norm is rough.

(i) $\Rightarrow$ (iii). By Theorem 1(i) $\Rightarrow$ (ii), there is an $\varepsilon' > 0$ such that for every $x \in S_X$ and $\delta > 0$, there exists $y \in S_X$, $\|y - x\| < \delta$ with $\mathrm{diam}(A(x) \cup A(y)) \geqslant \varepsilon'$. Let $f, g \in A(x) \cup A(y)$ such that $\|f - g\| > \varepsilon'/2$. Then there is an element $u \in S_X$ with $(f - g)(u) \geqslant \varepsilon'/2$. If $f \in A(x)$ and $g \in A(y)$ (similarly if $g \in A(x)$ and $f \in A(y)$), then by (1) we obtain $\varepsilon'/2 \leqslant f(u) + g(-u) \leqslant \tau(x, u) + \tau(y, -u)$. In a similar way, if $f, g \in A(x)$ or $f, g \in A(y)$ we obtain respectively $\tau(x, u) + \tau(x, -u) \geqslant \varepsilon'/2$ or $\tau(y, u) + \tau(y, -u) \geqslant \varepsilon'/2$. Therefore we have (iii) e.g., for $\varepsilon = \varepsilon'/2$.

Since (iii) $\Rightarrow$ (ii) is obvious, this completes the proof of the corollary.

Using a slight modification of [6, Lemma 2] we have

LEMMA 1. *Let $x, y, u \in S_X$. If $\tau(x, u) + \tau(y, -u) = \varepsilon > 0$, then there is a $v \in S_X$ such that $\tau(x, v) \geqslant \varepsilon/4$ and $\tau(y, -v) \geqslant (\varepsilon/4) - \|x - y\|/2$.*

PROOF. By (1) there are $f \in A(x)$, $g \in A(y)$ such that $(f - g)(u) = \varepsilon$. Let $w = u - (f + g)(u)x/2$. Then $0 < \|w\| \leqslant 2$. Let $v = w/\|w\|$. We have

$$\tau(x, v) \geqslant f(v) = \frac{1}{\|w\|} \left(\frac{(f - g)(u)}{2} \right) = \frac{\varepsilon}{2\|w\|} \geqslant \frac{\varepsilon}{4}.$$

Since $g \in A(y)$, we have $g(x) \geqslant 1 - \|x - y\|$ and so

$$g(-u) + \frac{(f + g)(u)}{2} g(x) \geqslant \frac{(f - g)(u)}{2} - \|x - y\|.$$

Hence

$$\tau(y, -v) \geqslant g(-v) = \frac{1}{\|w\|} \left(g(-u) + \frac{(f + g)(u)}{2} g(x) \right)$$

$$\geqslant \frac{1}{\|w\|} \left(\frac{\varepsilon}{2} - \|x - y\| \right) \geqslant \frac{\varepsilon}{4} - \frac{\|x - y\|}{2}.$$

An immediate consequence of Corollary 1(i) ⇔ (iii), Lemma 1, formula (1) and Theorem 1 is

COROLLARY 2. *The following properties of a given norm of X are equivalent*:
(i) $\|\cdot\|$ *is rough*.
(ii) *There is an $\varepsilon > 0$ such that for every $x \in S_X$ and $\delta > 0$ there exist $y, u \in S_X$, $\|y - x\| < \delta$, such that either $\tau(y, u) \geqslant \varepsilon$ and $\tau(y, -u) \geqslant \varepsilon$ or $\tau(x, u) \geqslant \varepsilon$ and $\tau(y, -u) \geqslant \varepsilon - \delta$.*

We recall [**10**, p. 976] that a normed linear space E has *property* ($\mathfrak{I}$) if each closed bounded convex set $K \subset E$ is the intersection of the closed balls containing it. We need the following result of [**10**, Lemma 4.1].

LEMMA 2 [**10**, p. 979]. *Suppose that the normed linear space E has property* ($\mathfrak{I}$). *If $f \in S_{E^*}$ and $0 < \varepsilon < 1$, there exist $\delta(\varepsilon) > 0$ and $x \in S_E$ such that $y \in S_E \cap B_E(x, \delta)$ implies $A(y) \subset B_{E^*}(f, \varepsilon)$.*

COROLLARY 3. *If X has property* ($\mathfrak{I}$), *then the norm of X is not rough.*

PROOF. Use Theorem 1 and Lemma 2.

Though Lemma 2 property characterizes property ($\mathfrak{I}$) [**5**, p. 114], the converse to Corollary 3 is not always true. Indeed, it is known [**12**, p. 178] that if a Banach space X is reflexive, then X is an Asplund space (i.e., every continuous convex function defined on an open convex subset of X is Fréchet differentiable on a dense G_δ subset of its domain); by [**7**, p. 344] X is an Asplund space if and only if X does not admit any equivalent rough norm. In particular, any norm on a 2-dimensional Banach space is not rough. On the other hand, by [**10**, p. 981] a 2-dimensional Banach space X has property ($\mathfrak{I}$) if and only if X is smooth.

Due to the terminology, the following result seems surprising. *There is a Banach space X whose norm is both rough and smooth.*

EXAMPLE. In [**10**, p. 981] a norm for $m = (l^1)^*$ (the space of bounded sequences $y = \{y_i\}$) was defined by $\|y\| = \sup |y_i| + (\Sigma y_i^2/2^i)^{1/2}$ and for $x = \{x_i\} \in l^1$, $\|x\| = \sup\{\Sigma x_i y_i \mid y = \{y_i\} \in m, \|y\| = 1\}$. Then $X = (l^1, \|\cdot\|)$ is a smooth space

[10, p. 981]. Let $\varepsilon < \frac{1}{8}$ and $x \in S_X$. It was shown in [10, p. 982] that for any $y \in S_{X^*}$ such that $(x, y) = 1$, there exists a sequence $\{y^k\} \subset X^*$ such that $(x, y^k) \to 1$, $\|y^k\| \to 1$ and $\|y^k - y\| \geq \frac{1}{8}$. (Note that in the first case in [10, p. 982] it was not used that $y_i \to 0$ but only that $|y_i| < \frac{1}{4}$ for $i > n$.) Now, for $f_k, g_k \in S_{X^*}$ defined by $f_k = y, g_k = y^k/\|y^k\|$, we have $\lim f_k(x) = \lim g_k(x) = 1$ and $\limsup\|f_k - g_k\| > \varepsilon$, whence by Proposition 1, the norm on X is rough.

We remark also that there is a Banach space having a nonrough norm with no point of Gateaux differentiability [7, p. 341].

The characterization given in Theorem 1, helps us to introduce the following concept dual to rough norm.

DEFINITION 2. A norm of a Banach space X is said to have *property* (dr) if there is an $\varepsilon > 0$ such that for every $f \in \mathrm{Dom}(A^{-1})$ and $\delta > 0$, there exists $g \in \mathrm{Dom}(A^{-1})$, $\|g - f\| < \delta$ with $\mathrm{diam}(A^{-1}(f) \cup A^{-1}(g)) \geq \varepsilon$. When the Banach space is a dual space X^* and in the above definition f and g are chosen to be w^*-continuous, then the dual norm of X^* is said to have *property* $(w^* dr)$.

To prove the next result we need the following extension of Bishop and Phelps theorem given in [2, Theorem 1].

THEOREM 2 [2, p. 181]. *Suppose* $x \in S_X, f \in S_{X^*}$ *and* $|f(x) - 1| \leq \varepsilon^2/2$ $(0 < \varepsilon < \frac{1}{2})$. *Then there exist* $y \in S_X$ *and* $g \in A(y)$ *such that* $\|f - g\| \leq \varepsilon$ *and* $\|x - y\| < \varepsilon + \varepsilon^2$.

THEOREM 3. *Let* X *be a Banach space. Then*:
(i) *The norm of* X *has property* (dr) *if and only if the dual norm of* X^* *is rough.*
(ii) *The norm of* X *is rough if and only if the dual norm of* X^* *has property* $(w^* dr)$.

PROOF. Suppose the norm of X has property (dr) and let $\varepsilon > 0$ be given by Definition 2. Let $f \in S_{X^*}$ and $\delta > 0$. By Bishop and Phelps theorem there exists $f_1 \in \mathrm{Dom}(A^{-1})$ such that $\|f_1 - f\| < \delta/2$. By property (dr) there exists $f_2 \in \mathrm{Dom}(A^{-1})$, $\|f_2 - f_1\| < \delta/2$ with $\mathrm{diam}(A^{-1}(f_1) \cup A^{-1}(f_2)) \geq \varepsilon$. Let $x, y \in A^{-1}(f_1) \cup A^{-1}(f_2)$ such that $\|x - y\| \geq \varepsilon/2$. Regarding X as a subspace of X^{**} we have for $F \in A(f)$ that $\varepsilon/2 \leq \|x - y\| \leq \|x - F\| + \|F - y\|$, and so we can suppose that $\|x - F\| \geq \varepsilon/4$. Since $\|f_i - f\| < \delta$, $i = 1, 2$ and $x \in A(f_i)$ for some $i = 1$ or 2, it follows that $\mathrm{diam}(A(f) \cup A(f_i)) \geq \varepsilon/4$, whence by Theorem 1, the dual norm of X^* is rough.

Conversely, suppose the dual norm of X^* is rough and let $0 < \varepsilon < 2$ be given by Proposition 1(ii). Let $f \in \mathrm{Dom}(A^{-1})$ and $0 < \delta < \varepsilon/4$. By Proposition 1, there exist $x_n, y_n \in S_X$, $\lim f(x_n) = \lim f(y_n) = 1$ such that $\limsup\|x_n - y_n\| > \varepsilon$. Let n_0 be such that $|1 - f(x_{n_0})| < \delta^2/8$, $|1 - f(y_{n_0})| < \delta^2/8$ and $\|x_{n_0} - y_{n_0}\| \geq \varepsilon$, and let $x \in A^{-1}(f)$. Then $\varepsilon \leq \|x_{n_0} - y_{n_0}\| \leq \|x_{n_0} - x\| + \|x - y_{n_0}\|$, and so we can suppose that $\|x_{n_0} - x\| \geq \varepsilon/2$. By Theorem 2, there exist $z_0 \in S_X$ and $f_0 \in A(z_0)$ such that $\|z_0 - x_{n_0}\| < \delta$ and $\|f - f_0\| < \delta$. Then $\varepsilon/2 \leq \|x_{n_0} - x\| \leq \|x_{n_0} - z_0\| + \|z_0 - x\| < (\varepsilon/4) + \mathrm{diam}(A^{-1}(f) \cup A^{-1}(f_0))$, and so $\mathrm{diam}(A^{-1}(f) \cup A^{-1}(f_0)) \geq \varepsilon/4$, i.e., the norm of X has property (dr).

Since (ii) is obvious, this completes the proof.

REMARK 2. By the above proof of (i) it follows that the norm of X has property (dr) if and only if there exists an $\varepsilon > 0$ such that for every $f \in \mathrm{Dom}(A^{-1})$,

$x \in A^{-1}(f)$ and $\delta > 0$, there exists $g \in \text{Dom}(A^{-1})$, $\|f - g\| < \delta$ with

$$\sup_{y \in A^{-1}(g)} \|x - y\| \geq \varepsilon.$$

There exists a strictly convex Banach space X whose norm has property (dr). Indeed, let $\|\cdot\|$ be the norm on l^1 defined in the above example. Then c_0 has an equivalent norm whose dual norm on $c_0^* = l^1$ is $\|\cdot\|$. Let X be c_0 endowed with this equivalent norm. Then X^* being smooth, X is strictly convex and the norm of X^* being rough, by Theorem 3(i), the norm of X has property (dr).

DEFINITION 3 [7, p. 337]. A norm of a Banach space X is said to be *strongly rough* if there is an $\varepsilon > 0$ such that for every $x \in S_X$ there is an element $u \in S_X$ with $\tau(x, u) + \tau(x, -u) \geq \varepsilon$.

REMARK 3. By [7, Proposition 2(i) $\Leftrightarrow$ (ii)] it follows that $\|\cdot\|$ is strongly rough if and only if there is an $\varepsilon > 0$ such that for every $x \in S_X$, diam $A(x) \geq \varepsilon$.

REMARK 4. With a proof similar with that of Corollary 2, one can show that the following properties of a given norm of X are equivalent:

(i) $\|\cdot\|$ is strongly rough.

(ii) There is an $\varepsilon > 0$ such that for every $x \in S_X$ there exists $u \in S_X$ with $\tau(x, u) \geq \varepsilon$ and $\tau(x, -u) \geq \varepsilon$.

(iii) There is an $\varepsilon > 0$ such that for every $x \in S_X$ there exist $f_1, f_2 \in A(x)$ with $\{z \in S_X | f_1(z) \geq \varepsilon\} \cap \{z \in S_X | f_2(z) \leq -\varepsilon\} \neq \varnothing$.

We want now to introduce a concept dual to strongly rough norm. As we shall see, this is far from being satisfactory.

DEFINITION 4. A norm of a Banach space X is said to have *property (dsr)* if there is an $\varepsilon > 0$ such that for every $f \in \text{Dom}(A^{-1})$, $\text{diam}(A^{-1}(f)) \geq \varepsilon$. When the Banach space is a dual space X^* and in the above definition f is chosen to be w^*-continuous, then the dual norm of X^* is said to have *property (w^* dsr)*.

It is obvious that the norm of X is strongly rough if and only if the dual norm of X^* has property (w^* dsr). A result similar to Theorem 3(i) is no longer true, since for example the space c_0 endowed with the usual norm has property (dsr) while the dual norm of c_0^* is not strongly rough. We do not have an example of a Banach space X with the following two properties: the norm of X has not the property (dsr) and the dual norm of X^* is strongly rough.

For each Banach space X, let $\varepsilon_X = \sup\{\varepsilon \,|\, \text{diam } A(x) \geq \varepsilon \text{ for each } x \in S_X\}$. Then clearly this supremum is attained, $0 \leq \varepsilon_X \leq 2$, and we have

$$\varepsilon_X = \inf\{\text{diam } A(x) \,|\, x \in S_X\}.$$

The known examples of spaces are with $\varepsilon_X = 0$ or 2; these are not the only cases as the next result shows.

THEOREM 4. *For each λ, $0 < \lambda < 2$, there exist Banach spaces X with $\varepsilon_X = \lambda$.*

PROOF. Let $E = l^1(S)$, S uncountable, endowed with the usual norm. Let $f_0 = (\alpha_s)_{s \in S} \in E^*$, where $\alpha_{s_0} = 1$ for some fixed $s_0 \in S$ and $\alpha_s = 0$ for $s \neq s_0$. Choose β such that $1 - (\lambda/2) < \beta < (2/\lambda) - (\lambda/2)$ and let C be the convex hull of B_{E^*}, $B_{E^*}(\beta f_0, \lambda/2)$ and $B_{E^*}(-\beta f_0, \lambda/2)$. Then C is a symmetric w^*-compact convex

body of E^*, and it induces an equivalent norm on E by $|||x||| = \sup\{f(x)\,|\,f \in C\}$ for each $x \in E$. We have $\|x\| \leqslant |||x||| \leqslant (\beta + \lambda/2)\|x\|$, $x \in E$. Let $X = (E, |||\cdot|||)$. Then for each $x \in S_X$, the set $A_X(x)$ intersects either the set $B_{E^*} \cup B_{E^*}(\beta f_0, \lambda/2)$ or $B_{E^*} \cup B_{E^*}(-\beta f_0, \lambda/2)$. Suppose $A_X(x)$ intersects $B_{E^*} \cup B_{E^*}(\beta f_0, \lambda/2)$ (the other case is similar). If $A_X(x) \cap B_{E^*} \neq \varnothing$ then $A_E(x) \subset A_X(x)$, and for $f_1, f_2 \in A_E(x)$ we have $2\|f_1 - f_2\|/(2\beta + \lambda) \leqslant |||f_1 - f_2|||$. Since $\operatorname{diam}_{E^*} A_E(x) = 2$, it follows $4/(2\beta + \lambda) \leqslant \operatorname{diam}_{X^*} A_X(x)$, whence by our assumption on β we have $\operatorname{diam}_{X^*} A_X(x) \geqslant \lambda$. If $A_X(x) \cap B_{E^*}(\beta f_0, \lambda/2) \neq \varnothing$ and $A_X(x) \cap B_{E^*} = \varnothing$, then $A_X(x) \subset B_{E^*}(\beta f_0, \lambda/2)$. Since for each $f = (\eta_s)_{s \in S} \in E^*$ with $\eta_{s_0} = 0$ we have $\|f\| = |||f|||$, it follows $\operatorname{diam}_{X^*} A_X(x) = \lambda$. Note that this last case happens e.g., for $x = (\xi_s)_{s \in S} \in S_X$ where $\xi_{s_0} = 2/(2\beta + \lambda)$ and $\xi_s = 0$ for $s \neq s_0$. Therefore we have $\varepsilon_X = \lambda$.

We conclude this note with some questions.

(1) Is it true that for any Banach space X with $0 < \varepsilon_X < 2$, there exists an equivalent norm for which the corresponding ε_X equals 2? When $\varepsilon_X = 0$ such a question has in general a negative answer, since [9, 7, p. 343] if there is an equivalent smooth norm, then there is no equivalent strongly rough norm. On the other hand, in [7, p. 346] the space $X = m$ (which has no equivalent smooth norm [3, p. 522]) has been renormed in such a way that $\varepsilon_X = 2$. So we can ask whether this is not a general phenomenon, i.e., whether for any Banach space X with no equivalent smooth norm, there is an equivalent norm such that the corresponding ε_X equals 2.

(2) There exist spaces X with the property that $\operatorname{diam} A(x) = \varepsilon_X$ for all $x \in S_X$. For example, $l^1(S)$, S uncountable, has this property with $\varepsilon_X = 2$, and the smooth spaces with $\varepsilon_X = 0$. Is there a Banach space X such that $\operatorname{diam} A(x) = \varepsilon_X$ for all $x \in S_X$ and $0 < \varepsilon_X < 2$? If the answer is affirmative, is it true that for any λ, $0 < \lambda < 2$, there exists a Banach space X such that $\varepsilon_X = \lambda$ and $\operatorname{diam} A(x) = \varepsilon_X$ for all $x \in S_X$?

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