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SYSTEMS OF DIFFERENTIAL EQUATIONS SUBJECT TO MILD INTEGRAL CONDITIONS

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ABSTRACT. It is shown that solutions of a system $x' = f(t, x)$ approach constant vectors as $t \to \infty$, under assumptions which do not require that $\|f(t, x)\| \leqslant w(t, \|x\|)$, where w is nondecreasing in $\|x\|$, and which permit some or all of the integral smallness conditions on f to be stated in terms of ordinary—rather than absolute—convergence. Estimates of the order of convergence are given.

Theorems which imply that solutions of a system

$$x' = f(t, x) \tag{1}$$

approach constant vectors as $t \to \infty$ are usually obtained by assuming that

$$\|f(t, x)\| \leqslant w(t, \|x\|), \tag{2}$$

where w is decreasing in r for $r \geqslant 0$, and subject to some integral condition such as

$$\int^{\infty} w(t, r)\, dt < \infty, \qquad r \geqslant 0; \tag{3}$$

for examples, see [**1, 4, 6**]. Sometimes, instead of (3), it is required that the equation $r' = w(t, r)$ have a bounded positive solution on some half-line $[t_0, \infty)$; for example, see [**2**]. Clearly, any integral condition like (3) requires absolute convergence, and there are systems for which no majorizing function w as in (2) is increasing in $\|x\|$. Here we present results which do not require this kind of bound on f, and are based on integral conditions involving ordinary—that is, not necessarily absolute—convergence of some of the improper integrals in question. Moreover, our results include information on the order of convergence of solutions of (1).

Throughout this paper the norm $\|\ \|$ of a vector or matrix is the sum of the absolute values of its elements, and W and K are as in the following assumption. The motivation for this definition will become clear after the statement of the main theorem (Remark 1, below).

Assumption A. The $n \times n$ matrix function W is continuously differentiable and invertible on $[\overline{T}, \infty)$ for some $\overline{T} \geqslant 0$, and

$$\int_t^{\infty} \|W(t)(W^{-1}(s))'\|\, ds \leqslant K < \infty, \qquad t \geqslant \overline{T}. \tag{4}$$

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Henceforth we assume that $t \geqslant \bar{T}$. Notice that

$$\int^{\infty} \|(W^{-1}(s))'\|\, ds < \infty \tag{5}$$

and so

$$\lim_{t\to\infty} W^{-1}(t) \text{ exists (finite)}, \tag{6}$$

because of (4).

LEMMA 1. *If q is a continuous n-vector function on $[T, \infty)$ and $\int^{\infty} W(t)q(t)\, dt$ converges, then $\int^{\infty} q(t)\, dt$ converges, and*

$$\left\| W(t)\int_t^{\infty} q(s)\, ds \right\| \leqslant (1 + K)\rho(t), \qquad t \geqslant T, \tag{7}$$

where

$$\rho(t) = \sup_{\lambda \geqslant t} \left\| \int_{\lambda}^{\infty} W(s)q(s)\, ds \right\|. \tag{8}$$

PROOF. With

$$p(t) = \int_t^{\infty} W(s)q(s)\, ds, \tag{9}$$

integration by parts yields

(10)

$$\int_t^{\bar{t}} q(s)\, ds = \int_t^{\bar{t}} W^{-1}(s)W(s)q(s)\, ds = -W^{-1}(s)p(s)\Big|_t^{\bar{t}} + \int_t^{\bar{t}} (W^{-1}(s))'p(s)\, ds.$$

From (6), (8), and (9),

$$\|(W^{-1}(s))'p(s)\| \leqslant \rho(T)\|(W^{-1}(s))'\|, \qquad s \geqslant T,$$

and $\lim_{\bar{t}\to\infty} W^{-1}(\bar{t})p(\bar{t}) = 0$; hence, because of (5), we can let $\bar{t} \to \infty$ in (10) and obtain

$$\int_t^{\infty} q(s)\, ds = W^{-1}(t)p(t) + \int_t^{\infty} (W^{-1}(s))'p(s)\, ds.$$

Multiplying by $W(t)$ and invoking (4), (8), and (9) yields (7).

DEFINITION 1. For $T \geqslant 0$, let $H(T)$ be the Banach space of continuous n-vector functions h on $[T, \infty)$ such that Wh is bounded, with the norm

$$N(T; h) = \sup_{t \geqslant T} \|W(t)h(t)\|. \tag{11}$$

If $M > 0$, let

$$H_M(T) = \{h \in H(T) \mid N(T; h) \leqslant M\}.$$

The following is our main theorem:

THEOREM 1. *For a given vector c, suppose there are constants $M > 0$ and $T_0 \geqslant 0$ such that f is continuous on*

$$\Omega = \{(t, x) \mid t \geqslant T_0,\ \|W(t)(x - c)\| \leqslant M\} \tag{12}$$

and the integral

$$J(t;h)=\int_t^\infty W(s)f(s,c+h(s))\,ds,\qquad t\geqslant T, \tag{13}$$

converges if $h\in H_M(T)$ *and* $T\geqslant T_0$. *Suppose also that*

$$\|J(t;h_1)-J(t;h_2)\|\leqslant \delta N(T;h_1-h_2),\qquad t\geqslant T, \tag{14}$$

whenever

$$h_1,h_2\in H_M(T)\quad \text{with } T\geqslant T_0, \tag{15}$$

where

$$0<\delta<1/(1+K). \tag{16}$$

Then (1) *has a solution* x_0 *which is defined for* t *sufficiently large and satisfies*

$$\lim_{t\to\infty} W(t)(x_0(t)-c)=0; \tag{17}$$

moreover, if x_1 *is any solution of* (1) *such that*

$$\lim_{t\to\infty} W(t)(x_1(t)-c)=0, \tag{18}$$

then

$$x_1(t)=x_2(t) \tag{19}$$

for t *sufficiently large.*

REMARK 1. In our examples, $W=\operatorname{diag}[w_1,\dots,w_n]$ with $w_i>0$ and $w_i'\geqslant 0$, in which case (4) holds with $K=n$, and (17) implies that the components of x_0 approach c at possibly different rates. We state and prove our theorems under the more general Assumption A because this does not complicate the proofs and the more general formulation may be useful in some applications.

PROOF OF THEOREM 1. If $h\in H_M(T)$ with $T\geqslant T_0$, then

$$\|J(t;h)\|\leqslant\|J(t;h)-J(t;0)\|+\|J(t;0)\|\leqslant\delta M+\|J(t;0)\|, \tag{20}$$

from (14) with $h_1=h$ and $h_2=0$. Now choose $T_1\geqslant T_0$ so that

$$\delta M+\sup_{t\geqslant T_1}\|J(t;0)\|\leqslant M/(1+K), \tag{21}$$

which is possible because of (16) and the convergence of $J(t;0)=\int_t^\infty W(s)f(s,c)\,ds$. From (20) and (21),

$$\|J(t;h)\|\leqslant M/(1+K)\quad \text{if } t\geqslant T_1 \text{ and } h\in B_M(T_1). \tag{22}$$

If $h\in H_M(T_1)$, define $\hat h=\mathfrak{T}h$ by

$$\hat h(t)=-\int_t^\infty f(s,c+h(s))\,ds,\qquad t\geqslant T_1.$$

From (13) and Lemma 1 with $q(t)=f(t,c+h(t))$, $\hat h$ is defined and satisfies the inequality

$$\|W(t)\hat h(t)\|\leqslant(1+K)\sup_{\lambda\geqslant t}\|J(t;h)\|,\qquad t\geqslant T_1.$$

From this and (22), $\|W(t)\hat h(t)\|\leqslant M$, $t\geqslant T_1$. Therefore, $\hat h\in H_M(T_1)$; that is, $\mathfrak{T}$ transforms $H_M(T_1)$ into itself.

Now suppose $h_i \in B_M(T_1)$ and $\hat{h}_i = \mathfrak{T} h_i$ $(i = 1, 2)$. Then Lemma 1 with

$$g(t) = f(t, c + h_1(t)) - f(t, c + h_2(t))$$

implies that

$$\| W(t)(\hat{h}_1(t) - \hat{h}_2(t)) \| \leq (1 + K) \| J(t; h_1) - J(t; h_2) \|, \qquad t \geq T_1,$$

and so, from (11) and (14) (with $T = T_1$),

$$N(T_1; \hat{h}_1 - \hat{h}_2) \leq \delta(1 + K) N(T_1; h_1 - h_2). \tag{23}$$

Hence, from (16), $\mathfrak{T}$ is a contraction mapping of $H_M(T_1)$ into itself, and therefore there is an h_0 in $H_M(T_1)$ such that $h_0 = \mathfrak{T} h_0$; that is,

$$h_0(t) = -\int_t^\infty f(s, c + h_0(s))\, ds, \qquad t \geq T_1.$$

From Lemma 1 with $q(t) = f(t, c + h_0(t))$, $\lim_{t\to\infty} W(t) h_0(t) = 0$. Therefore, the function $x_0 = c + h_0$ satisfies (1) and (17).

If x_1 satisfies (1) and (18), then $h_1 = x_1 - c$ is in $H_M(T_2)$ for some $T_2 \geq T_1$, and

$$h_1(t) - h_0(t) = \int_t^\infty [f(s, c + h_0(s)) - f(s, c + h_1(s))]\, ds, \qquad t \geq T_2.$$

By an argument like that which led to (23),

$$N(T_2; h_1 - h_0) \leq \delta(1 + K) N(T_2; h_1 - h_0),$$

which implies that $h_1(t) = h_0(t)$ for $t \geq T_2$, because of (16). This implies (19) for $t \geq T_2$, and completes the proof.

We now apply Theorem 1 to the system

$$x' = A(t)\psi(x) + g(t), \qquad t > 0. \tag{24}$$

First we need the following definition:

Definition 2. A vector c is a Lipschitz point of a vector function ψ if there are constants $\rho, \lambda > 0$ such that $\psi(x)$ is defined whenever

$$\| x - c \| \leq \rho \tag{25}$$

and

$$\| \psi(x_1) - \psi(x_2) \| \leq \lambda \| x_1 - x_2 \| \tag{26}$$

if $\| x_i - c \| \leq \rho$, $i = 1, 2$.

Theorem 2. *Suppose A is an $n \times m$ matrix function and g is an n-vector function, both continuous on* $[0, \infty)$, *and c is a Lipschitz point of the m-vector function ψ. Suppose also that*

$$\int^\infty W(t)[A(t)\psi(c) + g(t)]\, dt \tag{27}$$

converges and

$$\int^\infty \| W(t) A(t) \| \, \| W^{-1}(t) \| \, dt < \infty. \tag{28}$$

Then the conclusions of Theorem 1 *hold for* (24).

PROOF. Let

$$\sigma = \sup_{t \geq \bar{T}} \|W^{-1}(t)\|, \tag{29}$$

which is finite because of (6). Let δ be any number that satisfies (16), let λ be as in (26), and choose T_0 so that

$$\int_{T_0}^{\infty} \|W(s)A(s)\| \, \|W^{-1}(s)\| \, ds \leq \delta/\lambda, \tag{30}$$

which is possible because of (28). Henceforth, let $t \geq T_0$. Finally, let

$$M = \rho/\sigma, \tag{31}$$

with ρ as in (25). We will show that T_0 and M satisfy the requirements of Theorem 1, for

$$f(t, x) = A(t)\psi(x) + g(t). \tag{32}$$

We must first show that f is continuous on Ω as defined in (12). If $(t, x) \in \Omega$, then

$$\|x - c\| \leq \|W^{-1}(t)\| \, \|W(t)(x - c)\| \leq \sigma M = \rho,$$

because of (29) and (31). Since ψ is continuous for all x satisfying (25), while A and g are continuous on $[0, \infty)$, it follows that f is continuous on Ω, and $f(t, c + h(t))$ is continuous on $[T, \infty)$ if $h \in H_M(T)$ with $T \geq T_0$. Moreover, if (15) holds, then

$$\begin{aligned}\|\psi(c + h_1(t)) - \psi(c + h_2(t))\| &\leq \lambda \|h_1(t) - h_2(t)\| \\ &\leq \lambda \|W^{-1}(t)\| \, \|W(t)(h_1(t) - h_2(t))\| \\ &\leq \lambda \|W^{-1}(t)\| N(T; h_1 - h_2), \qquad t \geq T,\end{aligned}$$

where we have used (11) and (26). This and (30) imply that

$$\left\| \int_t^{\infty} W(s)A(s)[\psi(c + h_1(s)) - \psi(c + h_2(s))] \, ds \right\| \leq \delta N(T; h_1 - h_2), \qquad t \geq T. \tag{33}$$

With f as in (32), the functional J in (13) becomes

$$J(t; h) = \int_t^{\infty} W(s)[A(s)\psi(c + h(s)) + g(s)] \, ds.$$

From the convergence of (27), $J(t; 0)$ exists. This and the convergence of the integral in (33) with $h_1 = h$ and $h_2 = 0$ imply that $J(t; h)$ exists for all $h \in H_M(T)$, if $t \geq T \geq T_0$. Knowing this, we can conclude from (33) that (14) holds whenever (15) does. This completes the proof of Theorem 2.

Stronger results are available for a linear system

$$x' = A(t)x + g(t), \qquad t > 0. \tag{34}$$

We omit the proof of the following theorem, which is similar to that of Theorem 2.

THEOREM 3. *Suppose the $n \times n$ matrix function A and the n-vector function g are continuous on* $[0, \infty)$,

$$\int^{\infty} \|W(t)A(t)W^{-1}(t)\| \, dt < \infty, \tag{35}$$

and

$$\int^{\infty} W(t)[A(t)c + g(t)]\, dt \tag{36}$$

converges for a given constant vector c. Then (34) *has a unique solution* x_0 *which satisfies* (17).

The next theorem follows easily from this and elementary properties of linear systems.

THEOREM 4. *Suppose A and g are continuous on* $[0, \infty)$, (35) *holds, and* $\int^{\infty} W(t)A(t)\,dt$ *and* $\int^{\infty} W(t)g(t)\,dt$ *converge. Then* (34) *has a unique solution which satisfies* (17) *for any given constant vector c, and every solution of* (34) *satisfies* (17) *for some c.*

Theorem 4 with $W = I$ contains a result of Brauer [**1**, Lemma 2], who assumed that $\int^{\infty} \|A(t)\|\, dt < \infty$ and $\int^{\infty} \|g(t)\|\, dt < \infty$.

In the following examples, $c = \operatorname{col}[\alpha_1, \alpha_2]$ and x_1 and x_2 are the components of a solution vector x.

EXAMPLE 1. The system

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \frac{4 \sin t}{t^4(x_1 - x_2)^2} \begin{bmatrix} 1 & 2 & -t \\ t^2 & t & -1 \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{bmatrix} + \frac{\sin t}{t^3} \begin{bmatrix} 1 \\ 1 - t \end{bmatrix}, \qquad t > 0, \tag{37}$$

is of the form (24). If $W = \operatorname{diag}[t^q, t^{q-1}]$ with $q \geqslant 1$, then (28) holds. If $\alpha_1 \neq \alpha_2$, then c is a Lipschitz point of ψ in (37), and (27) converges if $q < 3$. Therefore, Theorem 2 implies that (37) has a solution such that

$$x_1(t) = \alpha_1 + o(t^{-q}), \qquad x_2(t) = \alpha_2 + o(t^{-q+1}) \tag{38}$$

for all $q < 3$, provided $\alpha_1 \neq \alpha_2$. If $\alpha_1 = -\alpha_2 \neq 0$, then (27) converges for $q < 4$, and so (38) holds for all $q < 4$.

EXAMPLE 2. For the linear system

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2t^{-2} \sin t & 3t^{-1} \\ t^{-3} & t^{-3} \sin t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6t^{-1} \\ -2t^{-3} \end{bmatrix}, \tag{39}$$

(35) holds if $W = \operatorname{diag}[t^q, t^{q+1}]$ with $q \geqslant 0$, and (36) converges if α_1 is arbitrary, $\alpha_2 = -2$, and $q < 1$. Hence, from Theorem 3, (39) has a solution such that

$$x_1(t) = \alpha_1 + o(t^{-q}), \qquad x_2(t) = -2 + o(t^{-q-1}) \tag{40}$$

for all $q < 1$. If $\alpha_1 = -\alpha_2 = 2$, then (36) converges if $q < 2$, and so (40) holds for all $q < 2$ if $\alpha_1 = 2$.

EXAMPLE 3. We now exhibit a system $x' = A(t)x$ whose solutions all tend rapidly to constant vectors, even though $\int^{\infty} \|A(t)\|\, dt = \infty$. To this end, we first observe that $\int^{\infty} t^{\nu} e^{\mu t} \sin(e^t)\, dt$ converges for all ν if $\mu < 1$; if $\mu = 0$, it converges conditionally if $\nu \geqslant -1$, absolutely if $\nu < -1$. (To see this, substitute $\tau = e^t$.) Now consider the system

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \sin(e^t) \begin{bmatrix} at^{-3/2} & bt^{-1/2} \\ ct^{-5/2} & dt^{-3/2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{41}$$

where a, b, c, d are constants and $b \neq 0$, so that $\int^\infty \|A(t)\|\,dt = \infty$, and let $W = \operatorname{diag}[e^{\mu t}, te^{\mu t}]$ with $0 \leqslant \mu < 1$. Here $\int^\infty W(t)A(t)\,dt$ converges and (35) holds; hence, Theorem 4 implies that if α_1 and α_2 are arbitrary, then (41) has a solution such that

$$x_1(t) = \alpha_1 + o(e^{-\mu t}), \qquad x_2 = \alpha_2 + o(t^{-1}e^{-\mu t})$$

for all $\mu < 1$.

EXAMPLE 4. Suppose $a, b, f \in C(0, \infty)$ and z_1 and z_2 are solutions of

$$z'' + a(t)z = 0, \qquad t > 0, \tag{42}$$

with wronskian $\omega \neq 0$. Then y satisfies

$$y'' + a(t)y = b(t)y + f(t), \qquad t > 0, \tag{43}$$

if and only if

$$y^{(r)} = x_1 z_1^{(r)} + x_2 z_2^{(r)}, \qquad r = 0, 1, \tag{44}$$

where

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \frac{b}{\omega}\begin{bmatrix} -z_1 z_2 & -z_2^2 \\ z_1^2 & z_1 z_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{f}{\omega}\begin{bmatrix} -z_2 \\ z_1 \end{bmatrix}. \tag{45}$$

Let α_1 and α_2 be given constants, define $g = b(\alpha_1 z_1 + \alpha_2 z_2) + f$, and suppose $w \in C(0, \infty)$ is such that $w > 0$, $w' \geqslant 0$, and the integrals $\int^\infty wgz_i\,dt$ $(i = 1, 2)$ converge. Suppose also that $\int^\infty |b|(|z_1| + |z_2|)^2\,dt < \infty$. Then, applying Theorem 3 with $W = wI$ to (45) shows that (43) has a solution which satisfies (44), with

$$x_i(t) = \alpha_i + o(1/w(t)), \qquad i = 1, 2.$$

The special case of this with $w = 1$ and $f = 0$ is known [**5**].

EXAMPLE 5. If (42) is nonoscillatory, we may choose z_1 and z_2 so that z_2/z_1 tends monotonically to infinity [**3**, p. 357]. Then it is useful to apply Theorem 3 to (45) with

$$W = \operatorname{diag}\left[(z_2/z_1)^q, (z_2/z_1)^{q+1}\right] \qquad (q \geqslant 0).$$

For example, if $a = -1$ in (42), so that (43) becomes

$$y'' - y = b(t)y + f(t), \qquad t > 0, \tag{46}$$

we take $z_1 = e^{-t}$ and $z_2 = e^t$ in (44) and (45), and

$$W = \operatorname{diag}[e^{2qt}, e^{2(q+1)t}] \qquad (q \geqslant 0).$$

Theorem 3 then implies that if $\int^\infty |b(t)|\,dt < \infty$ and α_1 and α_2 are constants such that

$$\int^\infty e^{2qt}\left[b(t)(\alpha_1 + \alpha_2 e^{2t}) + f(t)e^t\right] dt$$

converges, then (46) has a solution y such that

$$y^{(r)}(t) = (-1)^r \alpha_1 e^{-t} + \alpha_2 e^t + o(e^{-(2q+1)t}), \qquad r = 0, 1.$$

This implies several results previously obtained under stronger integral conditions.

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