L.\-Boundedness of a Certain Class of Multipliers Associated with Curves on the Plane. II

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Abstract. L.\-boundedness of some multiplier, almost constant along curves, is proved. The interval of p's depends on the rate of decaying along the above curves.

Introduction. We get positive results for some multiplier considered in [3] after introducing a "weight" factor not heavy enough to allow us to apply Marcinkiewicz's multiplier theorem. The method used is a "geometric decomposition" as used in [1] for Bochner-Riesz multipliers. It can be proved that the present results are sharp in a similar way to that used in [3] for "Schrödinger multiplier". For the sake of simplicity Theorem 2 is stated in a particular case, nevertheless, the method remains valid for a wider range of functions. I would like to thank Professor A. Cordoba for the orientation and advice he has given to me.

1. Let us summarize some version of the theorems in [1] for more general curves than the circumference.

Let \( \gamma(t) \) be an arc of a smooth curve in \( \mathbb{R}^2 \), \( t \in [t_0, t_1] \), such that its curvature is positive at any \( t \in [t_0, t_1] \).

Pick a \( \delta > 0 \) small enough for there to be a collar or strip around \( \gamma \) of width \( \delta \) such that the mapping,

\[
(x_1, x_2) \in \text{Collar}_{\delta} \gamma \to (s, t)
\]

where \( (x_1, x_2) = x = \gamma(t) + s\eta(\gamma(t)) \) and \( \eta \) denotes the normal of \( r \) at \( \gamma(t) \), is one-to-one in the collar and has derivatives of a suitable order.

Let \( \phi \) and \( \Psi \) be \( C_0^\infty \) functions s.t. \( \phi(0) = 1 \); \( \phi \) is supported on \( (-1, 1) \) and \( \Psi \) is supported in \( [t_0, t_1] \). Notice the function defined by

\[
m(x) = \begin{cases} 
\phi\left(\frac{s}{\delta}\right)\Psi(t) & \text{when } x = \gamma(t) + s\eta(\gamma(t)), \ x \in \text{Collar}_{\delta} \gamma, \\
0 & \text{otherwise}
\end{cases}
\]

is in \( C_0^\infty \).

Define the operator

\[
(T_sf)(\xi) = m(\xi)\hat{f}(\xi), \quad f \in C_0^\infty(\mathbb{R}^2).
\]

Let \( \kappa(t) \) be the curvature of \( \gamma \) at \( \gamma(t) \); consider \( \kappa_2 \leq \kappa(t) \leq \kappa_1 \) for \( t \in [t_0, t_1] \) and \( \kappa_2 > 0 \), then the following holds.
THEOREM 1.
\[ \| T_\delta f \|_4 \leq C \frac{\kappa_1}{\kappa_2} |1 + \log \delta \kappa_1|^{1/4} \| f \|_4 \quad \text{for every} \ f \in C_0^\infty(\mathbb{R}^2). \]

We refer to the proof given in [1] for the case of \( \gamma \) a circumference. For further details see [2].

LEMMA 2.
\[ \| T_\delta f \|_\infty \leq C L \kappa_1^{1/2} \delta^{-1/2} \| f \|_\infty \]
for any \( f \in L^\infty(\mathbb{R}^2) \), where \( L \) is the length of the curve in \( t \in [t_0, t_1] \).

PROOF. It suffices to see that
\[ \| \hat{m}(\cdot) \|_{L^1} \leq C L \kappa_1^{1/2} \delta^{-1/2}. \]

It can be reduced by dilations to the case \( m_1 \) when \( \kappa_1 = 1 \), length \( L' = L \kappa_1 \) and \( \delta' = \delta \kappa_1 \), for \( \| m_1 \|_{L^1} = \| \hat{m} \|_{L^1} = \delta \kappa_1 \).

Following the theorem in [1] let us decompose the arc of the curve into \( L \delta^{-1/2} \) pieces such that
(a) the length of each of them is \( \delta^{1/2} \);
(b) the angle between the normals at two points in the same piece is less than \( \delta^{1/2} \).

According to this partition of the curve we are going to break down the multiplier; take \( \{ \Psi_j \}_{j=1}^{L \delta^{-1/2}} \) to be a smooth partition of the unity in \( [0, L] \) constructed in the following way:
\[ \Psi_j(t) = \Psi_0(\delta^{-1/2} t) \]
where \( \Psi \in C_0^\infty(\mathbb{R}) \) is supported in \( [-1, 1] \) and its Schwarz's bounds are independent of \( \delta \).

We can suppose the curve is referred to arclength \( t \). Then \( m(\xi) = \phi(s/\delta) \Psi(t) = \sum \phi(s/\delta) \Psi_j(t) \Psi(t) \) and for saving notation let us call on \( \Psi_j(t) \Psi(t) = \Psi_j(t) \); \( m_j(\xi) = \phi(s/\delta) \Psi_j(t) \).

By (a) and (b), \( m_j(\xi) \) is supported in a rectangle of length \( 2 \delta^{1/2} \) and width \( 2 \delta \), whose direction is the tangent to the curve at \( t = j \delta^{1/2} \). Set \( T_j f(\xi) = m_j(\xi) \hat{f}(\xi) \).

Then
\[ T_f(x) = \sum_{j=1}^{L \delta^{-1/2}} T_j f(x). \]

Now, \( \hat{m}_j(x) = e^{2\pi i x \cdot s} K_j(\rho_j x) \), where \( x_j \) is the middle point of the \( j \)-piece of curve, \( \rho_j \)
is a rotation and \( K_j \) satisfies
\[ (1) |K_j(x_1, x_2)| \leq C_p \delta^{3/2}/|x_2\delta^{1/2}|^p \quad \text{for} \ p = 0, 1, 2, 3, \]
\[ (2) |K_j(x_1, x_2)| \leq C_q \delta^{3/2}/|x_1\delta|^q \quad \text{for} \ q = 0, 1, 2, 3, \]
where \( C_p \) and \( C_q \) are independent of \( j \).
(This is proved using integration by parts and properties on the support of $m_j$; see [1] in the case of the circumference.)

Then $\hat{m}(x) = \sum e^{2\pi i x \cdot \rho_j} K_j(\rho_j x)$ and

$$\int |\hat{m}(x)|^2 \, dx \leq \sum_{j=1}^{L^2} \int |K_j(\rho_j x)|^2 \, dx \leq C L^{2-1/2}.$$ 

The last inequality follows from (1) and (2) by taking a suitable decomposition of $\int_{\mathbb{R}^1} |K_j(x_1, x_2)| \, dx.$

2. The case of a multiplier along a curve. Let $\phi \in C_0^\infty$ be supported on $[-1, 1]$. Given $y = y(x)$ an increasing function with $y''(x) > 0$, and $\Psi$ a function in $C_0^\infty(\mathbb{R})$ supported in $[a, b]$, $a > 0$, let us consider

$$m(\xi_1, \xi_2) = \phi(\xi_2 - y(\xi_1)) \Psi(\xi_1).$$

Then

Theorem 1'.

$$\|T\phi\|_4 \leq C \log \gamma_1^{-1} \cdot \kappa_1^{1/4} \|\phi\|_4$$

where $\gamma_1(\xi) = m(\xi) \hat{\phi}(\xi)$, $\kappa_1$ is an upper bound of the curvature of the curve $y(x) = y$, $\gamma_1$ is a lower bound of $\gamma'(x)$ when $x \in [a, b]$, $c = C_1\kappa_1/\kappa_2$ is the ratio between the maximum and minimum curvatures on $[a, b]$ and $c_1$ depends only on Schwarz’s bounds of $\phi$ and $\Psi$.

Lemma 2'.

$$\|T\phi\|_\infty \leq C L \gamma_1^{-1/2} \kappa_1^{1/2} \|\phi\|_\infty$$

where $L$ is the length of the arc of the curve $y = y(x)$, $x \in [a, b]$ and $C$ depends only on Schwarz’s bounds of $\phi$ and $\Psi$.

The proofs are as in Theorem 1 and Lemma 2; note that $\gamma_1^{-1}$ plays the role of $\delta$ there.

Theorem 2. Let

$$m(\xi_1, \xi_2) = \frac{1}{1 + (\xi_2 - \xi_1)^2} \phi(\xi_1, \xi_2)$$

be a bounded function, where $n$ is an integer $n \geq 2$ and $\phi(\xi_1, \xi_2)$ a $C^\infty$ function such that

$$\phi(\xi_1, \xi_2) = \frac{1}{|((\xi_1, \xi_2))|^2} \quad \text{when} \quad |(\xi_1, \xi_2)| > N, \quad \frac{1}{2} \geq \alpha > 0.$$ 

Then $m$ is a multiplier on $L^p$ for

$$\frac{4}{3 + 2\alpha} < p < \frac{4}{1 - 2\alpha}.$$ 

Proof. Let us take a partition of unity on $(0, \infty)$, $\{\psi_j\}_j=0^\infty$ such that $\psi_j(x) = \psi_j((x - 2^j)/2^{j-1})$ and $\text{supp} \psi_j \subset [0, 6]$. (We can do the same for $x < 0$.)
Let us split down

\[ m(\xi_1, \xi_2) = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} m(\xi_1, \xi_2) \Psi_l(\xi_2 - \xi_1^n) \Psi_k(\xi_1). \]

Set

\[ m_{l,k} = m(\xi_1, \xi_2) \cdot \Psi_l(\xi_2 - \xi_1^n) \Psi_k(\xi_1). \]

Except for a finite number of \( l \)'s and \( k \)'s depending only on \( N \),

\[ m_{l,k}(\xi_1, \xi_2) = \frac{1}{1 + (\xi_2 - \xi_1^n)^2} \cdot \frac{1}{(\xi_1^n + \xi_2^n)^{n/2}} \Psi_l \left( \frac{\xi_2 - \xi_1^n - 2l}{2l-1} \right) \Psi_k \left( \frac{\xi_1 - 2k}{2l-1} \right). \]

After a change of variables, taking the new variables \( (\xi_2 - 2^l)/2^l-1, \xi_1/2^{(l-1)/n} \), the norm of \( m_{l,k} \) as a multiplier on \( L^p \) is the same as the norm of

\[ m^l_{k,r}(\xi_1, \xi_2) = \frac{1}{2^{l(a+2)}} \frac{1}{2^{-2l} + (\xi_2 - \xi_1^n)^2 + 4(\xi_2 - \xi_1^n) + 4} \cdot \Psi_l(\xi_2 - \xi_1^n) \Psi_k(\xi_1 2^{l/n} - 2) \frac{1}{(\xi_1 2^{2l/n-2l} + \xi_2 + 4\xi_2 + 4)^{a/2}} \]

which is the product in the case \( kn > l \) of

\[ m^l_{k,r}(\xi_1, \xi_2) \]

\[ = \frac{1}{2^{l(a+2)}} \frac{1}{2^{-2l} + (\xi_2 - \xi_1^n)^2 + 4(\xi_2 - \xi_1^n) + 4} \cdot \Psi_l(\xi_2 - \xi_1^n) \Psi_k(\xi_1 2^{l/n} - 2) \frac{1}{(\xi_1 2^{2l/n-2l} + \xi_2 + 4\xi_2 + 4)^{a/2}} \]

and

\[ m^2_{l,k} = \chi_{[2^{a-n-1}, 2^{a-n-1} + 1]}(\xi_2) \cdot \chi_{[2^{a-n-1}, 2^{a-n-1} + 1]}(\xi_1) \]

Looking at (2.2) we notice that the function whose argument is \( (\xi_2 - \xi_1^n) \) has Schwarz's norms bounded independently of \( l \) and \( k \), and the support of \( \Psi_l(\xi_2, 2^{l/n} - 2) \) is contained in \( [2^{k-l/n-1}, 2^{k-l/n+2}] \). Then it is possible to apply Theorem 1' and to bound its \( (L^4, L^4) \)-norm by

\[ \frac{C}{2^{l(a+2)}} (\log | \gamma_l(k) |)^{1/4} \leq \frac{C}{2^{l(a+2)}} k^{1/4}. \]

In order to study \( m^2_{l,k} \), let us take \( \phi_{k,l} \) and \( \Omega_{k,l} \) in \( C_0^\infty(\mathbb{R}) \), such that

\[ \text{supp} \phi_{k,l} \subset [2^{kn-l/n-1}, 2^{kn-l/n+3n+2}] \]

and \( \phi_{k,l} \equiv 1 \) on \( [2^{kn-l/n+1}, 2^{kn-l/n+3n+1}] \); \( \text{supp} \Omega_{k,l} \subset [2^{k-l/n}, 2^{k-l/n+4}] \) and \( \phi_{k,l} \equiv 1 \) on \( [2^{k-l/n+1}, 2^{k-l/n+3}] \). Namely, \( \phi_{k,l}(x) = \phi(x/2^{k-l/n}) \) where \( \text{supp} \phi_0 \subset [\frac{1}{2}, 2^{3n} + 3] \), then \( m^2_{l,k} \) is the product of

\[ \chi_{[2^{a-n}-1, 2^{a-n}+1]}(\xi_2) \cdot \chi_{[2^{a-n}-1, 2^{a-n}+1]}(\xi_1) \]
and
\[ \phi_{k,l}(\xi_2) : \Omega_{k,l}(\xi_1) \frac{1}{|\xi_1^2 \cdot 2^{l/n-1/2} + \xi_2^2 + 4\xi_2 + 4 |^{n/2}}. \]

The first term of the product corresponds to a \((p, p)\) operator with constant norm independent of \(k\) and \(l\). The second one, after changing the variable \(\xi_2\) by \(\xi_2/2^{kn-l+n}\), has the same norm as
\[ \left(2^{k-1/2}/n + 2n\right) + \xi_2^2/2^{kn-l+n} + 4/2^{kn-2l+n} |^{2/3}/2 \]
which, as operator \(L^p \rightarrow L^p\), \(1 < p \leq \infty\), is bounded by \(c/2^{(kn-l+n)/2}\). Therefore,
\[ (2.5) \quad \| m_{l,k} \|_{L^p, L^p} \leq \frac{c}{2^{kn} \cdot 2^{2l}} \cdot k^{1/4}. \]

In the case \(kn \leq 1\), we decompose (2.1) in the product of
\[ m_{l,k}^1(\xi_1, \xi_2) = \frac{1}{2^{l(a+2)}} \Psi_1(\xi_2 - \xi_1) \xi_1(\xi_1^2 - 2) \]
and
\[ m_{l,k}^2(\xi_1, \xi_2) = \Psi_1(\xi_1^2/n - k - 2) = \frac{\chi_{[0,2]}(\xi_2)}{(\xi_1^2/2^{2l/n-2l} + \xi_2^2/2^{kn-2l} + 4\xi_2 + 4)^{\alpha/2}}. \]

The first one has \((p, p)\)-norm, \(1 < p < \infty\), as \(1/2^{l(a+2)}\); the second one has the same \((p, p)\)-norm as the multiplier
\[ (2.6) \quad \Psi(\xi_1) \frac{1}{2^{l(k-1)}} \xi_1^2/2^{2k-2l} + 4\xi_2/2^{2k-2l} + 4/2^{2k-2l} + 4 |^{\alpha/2} \]
obtained by the change \(\xi_1 \rightarrow (\xi_1 - 2^{k-1/n+1})/2^{k-1}/n\). The Schwarz's norm in (2.6), divided by \(1/2^{l(k-1)}\), is independent of \(k\) and \(l\) and we can see, in the way we did in (2.4) above, that \(\| m_{l,k}^2 \|_{L^p, L^p} \leq c/2^{(kn-l+n)/2}\) and then \(\| m_{l,k} \|_{L^p, L^p} \leq c/(2^k a 2^{l/2})\) in the case \(kn \leq l\). Therefore the sum \(\Sigma_{k,l, kn \leq l} \| m_{l,k} \|_{L^p, L^p}\) is convergent and it suffices to consider \(m_{l,k}\) with \(kn > l\). By Lemma 2',
\[ (2.7) \quad \| m_{l,k} \|_{\infty, \infty} \leq \frac{c}{2^{(a+2)} \cdot 2^{kn}/2}. \]

So, since (2.4) and (2.7) hold, we deduce that
\[ (2.8) \quad \| m_{l,k} \|_{\infty, \infty} \leq \frac{c}{2^{2l/2} \cdot 2^{kn/2}}. \]
Interpolation between \((L^\infty, L^\infty)\) boundedness arising from (2.8) and \((L^4, L^4)\) boundedness from (2.5) implies that

\[
\|m_{l,k}\|_{L^p, L^p} \leq \frac{c \cdot k^{1/4}}{2^{1/2} k^{n-1} / (n+1) + kn/p} \]

converges if \(p < 4/(1 - 2\alpha)\), and the result follows by duality and interpolation.

Note. The method of Theorem 2 could be applied to some other operators, as we noticed; one of them is

\[
m(\xi_1, \xi_2) = \frac{1}{(\xi_2 - \xi_1^n)^2 + 1} \left( |\xi_1|^2 + 1 \right)^{\alpha/2}
\]

It is possible to prove that \(m(\xi_1, \xi_2)\) is a multiplier on \(L^p\), when

\[
\frac{4}{3 + 2^{\alpha/n}} < p < \frac{4}{1 - 2^{\alpha/n}}.
\]

Following the method of [3] we could prove that the above mentioned intervals are the best possible; hence, if we set

\[
m(\xi_1, \xi_2) = \frac{1}{(\xi_2 - \xi_1^n)^2 + 1} \left( |\xi_1|^2 + 1 \right)^{\alpha/2},
\]

\(m(\xi_1, \xi_2)\) is a multiplier on \(L^p\) in an interval of \(p\)'s, neither (2.2) nor (1. \(\infty\)) taking \(n\) large enough. This multiplier arises from a partial differential equation on the plane.

References

2. A. Ruiz, Multiplicadores asociados a curvas en el plano y teoremas de restricción de la transformada de Fourier a curvas en \(\mathbb{R}^2\) y \(\mathbb{R}^3\), Tesis doctoral, Universidad Complutense de Madrid, 1980.

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