LENGTH OF RAY-IMAGES UNDER CONFORMAL MAPS

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Abstract. Let \( w = f(z) \) be regular and univalent in \(|z| < 1\) with \( f(0) = 0 \). Suppose that \( f \) maps the unit disc onto a domain \( D \). Let \( l(r, \theta) \) be the length of the image curve of the ray joining \( z = 0 \) to \( z = re^{i\theta} \) in \( D \) and \( A(r) = \sup \{ |f(re^{i\theta})|^{-1} l(r, \theta) \} \) where the supremum is taken over all starlike functions. In this paper we show that \( A(r) \leq (1 + r) \).

Let \( w = f(z) \) be regular and univalent in \(|z| < 1\) and \( f(0) = 0 \). Suppose that \( f \) maps the unit disc \( \{z: |z| < 1\} \) onto a domain \( D \). Let \( C(r, \theta) \) be the image in \( D \) of the ray joining \( z = 0 \) to \( z = re^{i\theta} \) and let

\[
l(r, \theta) = \int_0^r |f'(ue^{i\theta})| \, du
\]

be its length. Gehring and Hayman [1] proved that if \( D \) is starlike with respect to \( w = 0 \)

\[
l(r, \theta) < k |f(re^{i\theta})|
\]

where \( k \) is an absolute constant. Sheil-Small [4] showed that \( k \) can be taken as \((1 + \log 4)\) and if in addition \( f \) is starlike of order \( 1/2 \) then \( k \) can be taken as \((1 + \log 2)\). His conjecture regarding the best possible constants in both these cases has been proved in the affirmative by Hall [2,3]. In the later work Hall poses the problem of finding

\[
A(r, \alpha) = \sup \left\{ |f(re^{i\theta})|^{-1} \int_0^r |f'(ue^{i\theta})| \, du \right\}
\]

where the supremum is taken over all functions \( f \) which are starlike of order \( \alpha \) and \( \theta \) real. In this paper we show that \( A(r, 0) \leq (1 + r) \) and it is likely that this is best possible up to rotations even though I am not able to establish this fact at present.

Theorem. Let \( f \) be starlike of order 0 in the unit disc then

\[
l(r, \theta) = \int_0^r |f'(ue^{i\theta})| \, du \leq (1 + r) |f(re^{i\theta})|.
\]

Proof. Without loss of generality let us suppose \( \theta = 0 \). Since \( f \) is starlike of order 0 we have

\[
H(z) = \frac{zf'(zr)}{f(zr)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 + zre^{-it}}{1 - zre^{-it}} \, d\theta
\]

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where \( V(t) \) is increasing, \( V(-\pi) + \pi = V(\pi) - \pi \). Let us put \( h(z) = f(rz) \) so that \( H(z) = zh'(z)/h(z) \). Now we can write

\[
\frac{1}{u} \text{Re} H(u) = \frac{\partial}{\partial u} \log |h(u)| = \int_0^\pi \frac{1 - u^2 r^2}{ur(1 - 2ur \cos t + u^2 r^2)} dW(t)
\]

where \( dW = dV(t) - dV(-t) \) is nonnegative and has total mass 1. We have to show that

\[
\int_0^1 |h'(u)| \, du \leq (1 + r) |h(1)|.
\]

We write

\[
\int_0^1 |h'(u)| \, du = \int_0^1 |H(u)||h(u)| u^{-1} \, du
\]

\[
= \int_0^1 \text{Re} H(u)|h(u)| u^{-1} \, du + \int_0^1 (|H(u)| - \text{Re} H(u))|h(u)| u^{-1} \, du
\]

and observe that the first integral on the right side is \(|h(1)|\) by (2). Further from (1) and (2) we have

\[
|H(u)| - \text{Re} H(u)
\]

\[
\leq \int_0^\pi \left\{ \frac{1 + ur}{\sqrt{(1 + u^2 r^2 - 2ur \cos t)}} - \frac{1 - u^2 r^2}{(1 + u^2 r^2 - 2ur \cos t)} \right\} dW(t)
\]

and (2) gives

\[
\log |h(u)/h(1)| = \int_1^u \frac{\partial}{\partial u} \log |h(u)| \, du
\]

\[
= \int_0^\pi \log \left( \frac{u(1 + r^2 - 2r \cos t)}{(1 + u^2 r^2 - 2ur \cos t)} \right) dW(t)
\]

and so using Jensen’s inequality [5, p. 24] we can get

\[
\left| \frac{h(u)}{u} \right| \leq |h(1)| \int_0^\pi \frac{(1 + r^2 - 2r \cos t)}{(1 + u^2 r^2 - 2ur \cos t)} dW(t).
\]

Hence

\[
\int_0^1 (|H(u)| - \text{Re} H(u))|h(u)| u^{-1} \, du
\]

\[
\leq |h(1)| \int_0^\pi \int_0^\pi I(t, s) \, dW(s) \, dW(t)
\]

\[
\leq \frac{1}{2} |h(1)| \int_0^\pi \int_0^\pi (I(t, s) + I(s, t)) \, dW(s) \, dW(t)
\]

where

\[
I(t, s) = \int_0^1 \left\{ \frac{1 + ur}{\sqrt{(1 + u^2 r^2 - 2ur \cos t)}} - \frac{1 - u^2 r^2}{(1 + u^2 r^2 - 2ur \cos t)} \right\}
\]

\[
\cdot \frac{(1 + r^2 - 2r \cos s)}{(1 + u^2 r^2 - 2ur \cos s)} \, du.
\]
In order to prove (3) it suffices to show
\[ \int_0^\pi \int_0^\pi \left( I(t, s) + I(s, t) \right) dW(s) dW(t) < 2r. \]
Since \( dW \) is a probability measure, moreover \( I \) is even in both variables it will be sufficient for our result to show
\[ I(s, t) + I(t, s) < 2r \quad \text{for } 0 < s < t < \pi \]
(see [3]). In the above integral in (8) we change \( ur \) as \( u \) and we get
\[ I(t, s) = \frac{1}{r} \int_0^r \left\{ \frac{1 + u}{\sqrt{(1 + u^2 - 2u \cos t)}} - \frac{1 - u^2}{(1 + u^2 - 2u \cos t)} \right\} \frac{(1 + r^2 - 2r \cos s)}{(1 + u^2 - 2u \cos s)} \, du. \]
We observe that
\[ I(t, s) = \frac{1 + r^2 - 2r \cos s}{r} \left( J(t, s) - K(t, s) \right) \]
where
\[ J(t, s) = \int_0^r \frac{(1 + u)}{\sqrt{(1 + u^2 - 2u \cos t)}}(1 + u^2 - 2u \cos s) \, du \]
and
\[ K(t, s) = \int_0^r \frac{(1 - u^2)}{(1 + u^2 - 2u \cos t)(1 + u^2 - 2u \cos s)} \, du. \]
We evaluate these integrals. In (11) we change the variable \( u \) to
\[ y = \frac{(1 - u)}{\sqrt{(1 + u^2 - 2u \cos t)}} \]
and use \( S = \sin(s/2) \) and \( T = \sin(t/2) \) to get
\[ J(t, s) = \frac{1}{2S\sqrt{(T^2 - S^2)}} \left[ \arctan(T^2 - S^2/S) - \arctan(\sqrt{(T^2 - S^2)(1 - r)}/S\sqrt{(1 + r^2 - 2r \cos t)}) \right]. \]
Now for simplification purposes we write
\[ k = \sqrt{(T^2 - S^2)}/T, \]
\[ 1 - x^2 = (1 - r)^2/(1 - r)^2 + 4rT^2, \quad 0 < x \leq 2\sqrt{r}/1 + r < 1. \]
The relations (14) imply that
\[ (1 - r)^2/(1 - r)^2 + 4rS^2 = (1 - x^2)/(1 - k^2x^2), \]
\[ T^2 = (1 - r)^2x^2/4r(1 - x^2), \quad S^2 = (1 - r)^2(1 - k^2)x^2/4r(1 - x^2). \]
The relations (14) and (15) show that
\begin{equation}
1 + r^2 - 2r \cos s \frac{J(t, s)}{r} = \frac{2(1 - k^2x^2)}{kx^2\sqrt{1 - k^2}} \left( \arctan \left( \frac{k}{\sqrt{1 - k^2}} \right) - \arctan \left( \frac{k\sqrt{1 - x^2}}{\sqrt{1 - k^2}} \right) \sqrt{1 - k^2} \right) \\
= \frac{2(1 - k^2x^2)}{x^2(1 - k^2)} \left( \int_0^x \frac{(1 - k^2)t}{(1 - k^2t^2)\sqrt{1 - t^2}} \, dt \right) \\
= \frac{2(1 - k^2x^2)}{x^2(1 - k^2)} \left( \int_0^x \frac{t}{\sqrt{1 - t^2}} \, dt - \sum_{n=1}^{\infty} k^{2n} \int_0^x t^{2n-1} \sqrt{1 - t^2} \, dt \right) \\
K(t, s) = \frac{1}{4(T^2 - S^2)} \log \left( \frac{(1 - r)^2 + 4rT^2}{(1 - r)^2 + 4rS^2} \right)
\end{equation}

and
\begin{equation}
\frac{1 + r^2 - 2r \cos s}{r} K(t, s) = -\frac{(1 - k^2x^2)}{k^2x^2} \log(1 - k^2x^2).
\end{equation}

Interchanging the roles of \( s \) and \( t \) we can also get
\begin{equation}
J(s, t) = \frac{1}{2T\sqrt{(T^2 - S^2)}} \cdot \log \left[ \frac{T + \sqrt{T^2 - S^2}}{S} \frac{T - \sqrt{T^2 - S^2}\sqrt{1 - x^2}/\sqrt{1 - k^2x^2}}{\sqrt{T^2 - (T^2 - S^2)(1 - x^2)/1 - k^2x^2}} \right]
\end{equation}

and
\begin{equation}
\frac{1 + r^2 - 2r \cos t}{r} J(s, t) = \frac{2}{kx^2} \log \frac{1 + k}{\sqrt{1 - k^2x^2} + k\sqrt{1 - x^2}} \\
= \frac{2}{x^2} \int_0^x \frac{t}{\sqrt{(1 - t^2)\sqrt{1 - k^2t^2}}} \, dt \\
= \frac{2}{x^2} \sum_{n=0}^{\infty} C_n k^{2n} \int_0^x t^{2n+1} (1 - t^2)^{-1/2} \, dt
\end{equation}

where
\begin{equation}
C_n = \Gamma(n + 1/2)/\Gamma(1/2)\Gamma(n + 1).
\end{equation}

Now \( K(s, t) = K(t, s) \) and so
\begin{equation}
\frac{1 + r^2 - 2r \cos t}{r} K(s, t) = -\frac{1}{k^2x^2} \log(1 - k^2x^2).
\end{equation}
Thus using (16), (17), (18) and (20) we get

\[
I(t, s) + I(s, t) = \frac{2(1 - k^2 x^2)}{(1 - k^2)x^2} \left\{ \int_0^x \frac{t}{\sqrt{1 - t^2}} \, dt - \sum_{n=1}^{\infty} k^{2n} \int_0^x t^{2n-1} \sqrt{1 - t^2} \, dt \right\} + \frac{2}{x^2} \sum_{n=0}^{\infty} C_n k^{2n} \int_0^x \frac{t^{2n+1}}{\sqrt{1 - t^2}} \, dt + \left( \frac{2 - k^2 x^2}{k^2 x^2} \right) \log \left( 1 - k^2 x^2 \right).
\]

If we write \(I(t, s) + I(s, t)\) as a power series in \(k\) of the form \(\sum_{n=0}^{\infty} \Phi_n(x) k^{2n}\) we find after some work that

\[
\Phi_n(x) = -2 \int_0^x \frac{t^{2n-1}}{\sqrt{1 - t^2}} \, dt + \frac{2}{x^2} \sum_{n=0}^{\infty} C_n \int_0^x \frac{t^{2n+1}}{\sqrt{1 - t^2}} \, dt - \frac{2 x^{2n}}{n + 1} + \frac{x^{2n}}{n}.
\]

Expanding \(\Phi_n(x)\) also in a power series we find that

\[
\Phi_n(x) = \sum_{j=0}^{\infty} C_j \left[ \frac{1 + C_n}{j + n + 1} - \frac{1}{j + n} \right] x^{2j+2n} - \frac{2 x^{2n}}{n + 1} + \frac{x^{2n}}{n}.
\]

From (23) it is clear that \(\Phi_n(0) = 0\) and that the coefficient of \(x^{2j+2n}\) \((j \geq 1)\) is \(((n + j)C_n - 1)C_j\). Now

\[
((n + j)C_n - 1)C_j \geq ((n + 1)C_n - 1)C_j
\]

and \((n + 1)C_n\) can be shown by induction to be greater than or equal to 1 using (19). Hence the coefficients are positive and in particular \(\Phi_n(x)\) has at most one stationary value which in this case must be a minimum. Hence

\[
\Phi_n(x) \leq \text{Max}(\Phi_n(0), \Phi_n(x_0))
\]

where \(x_0\) is the maximum possible value for \(x\) namely \(x_0 = 2\sqrt{r/(1 + r)}\). However

\[
\Phi_n(x_0) \leq \text{Max}(\Phi_n(0), \Phi_n(1))
\]

since \(x_0\) may be less or greater than the value of \(x\) at which \(\Phi_n(x)\) is minimum. But from (22)

\[
\Phi_n(1) = -\frac{\Gamma(n)\Gamma(1/2)}{\Gamma(n + 1/2)} + \frac{n\Gamma(n)\Gamma(1/2)}{(n + (1/2))(n + (1/2))} + \frac{2}{2n + 1} - \frac{2}{n + 1} + \frac{1}{n}
\]

\[
= \frac{1}{2n + 1} \left\{ 2 - \frac{\Gamma(n)\Gamma(1/2)}{\Gamma(n + (1/2))} \right\} - \frac{2}{n + 1} + \frac{1}{n}.
\]

Since \((n + 1/4)\Gamma^2(n + 1/2)/\Gamma^2(n + 1)\) increases to 1 we have

\[
\Gamma(n + 1/2)/\Gamma(n + 1) \leq 2/\sqrt{(1 + 4n)}.
\]

Using (24) and the fact that \(\pi > 2\) one can easily show that \(\Phi_n(1) < 0\) for \(n \geq 5\). For \(n < 5\) one can check this inequality directly. Hence \(\Phi_n(x) < 0\). For all \(n \geq 1\). Thus we get

\[
I(s, t) + I(t, s) \leq \Phi_0(x) = \frac{4}{x^2} - 4 \sqrt{(1 - x^2)} - 2.
\]
Now
\[ \frac{d}{dx} (\Phi_0(x)) = \frac{-8}{x^3} + \frac{4(2 - x^2)}{x^3 \sqrt{1 - x^2}} > 0. \]

Hence \( \Phi_0(x) \) increases and has a maximum at \( x = x_0 \). Thus
\[ (26) \quad \Phi_0(x) = \Phi_0(x_0) = 2r. \]

We use (26) in (25) and conclude from (7) that
\[ (27) \quad \int_0^1 \left[ |H(u)| - \Re H(u) \right] \frac{h(u)}{u} \, du \leq r \, |h(1)| \]

and as already observed
\[ (28) \quad \int_0^1 \Re H(u) |h(u)| u^{-1} \, du = |h(1)|. \]

Now (27) and (28) and (4) gives (3). Hence the result.

REFERENCES
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