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SPATIAL THEORY FOR ALGEBRAS OF UNBOUNDED OPERATORS. II

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ABSTRACT. In the previous paper [6], we have studied the spatial theory of O_p^*-algebras with a strongly cyclic vector. In this paper, we will investigate the spatial theory between O_p^*-algebras induced by a positive invariant sesquilinear form, which contains the former result.

1. Introduction. The spatial theory of von Neumann algebras has been investigated in detail, but the study for unbounded operator algebras seems to be hardly done except in [**6** and **7**].

In this paper we will continue the study in a more general framework which contains the former result [**6**].

In §2, we will introduce some notions which are used in this paper, for example, O_p^*-algebra, positive invariant sesquilinear form and so on.

In §3, we will consider a generalization of strongly cyclic vector for an O_p^*-algebra, which we call a strongly cyclic vector representation.

In §4, we will investigate the spatial theory between O_p^*-algebras induced by a positive invariant sesquilinear form.

2. Preliminaries. Let $\mathfrak{D}$ be a dense subspace of a Hilbert space $\mathcal{H}$. By $\mathfrak{L}(\mathfrak{D})$, we denote the set of all closed operators of $\mathfrak{D}$ into $\mathfrak{D}$. By $\mathfrak{L}^+(\mathfrak{D})$ we denote the set of all $A \in \mathfrak{L}(\mathfrak{D})$ with $\mathfrak{D}(A^*) \supset \mathfrak{D}$ and $A^*\mathfrak{D} \subset \mathfrak{D}$, where $\mathfrak{D}(A^*)$ denotes the domain of the adjoint operator A^*. When we put $A^+ = A^*/\mathfrak{D}$, the map $A \in \mathfrak{L}^+(\mathfrak{D}) \to A^+ \in \mathfrak{L}^+(\mathfrak{D})$ becomes an involution, and $\mathfrak{L}^+(\mathfrak{D})$ is a $*$-algebra with the usual operation. For a subalgebra $\mathfrak{A}$ of $\mathfrak{L}(\mathfrak{D})$, $(\mathfrak{A}, \mathfrak{D})$ is called an O_p-algebra and for a $*$-subalgebra $\mathfrak{A}$ of $\mathfrak{L}^+(\mathfrak{D})$, $(\mathfrak{A}, \mathfrak{D})$ is called an O_p^*-algebra. We note that an O_p^*-algebra is an O_p-algebra.

Let $(\mathfrak{A}, \mathfrak{D})$ be an O_p-algebra. Then we define seminorms $\|\cdot\|_A$ on $\mathfrak{D}$ by $\|\xi\|_A = \|A\xi\|$ for $A \in \mathfrak{A}_I$ and $\xi \in \mathfrak{D}$, where $\mathfrak{A}_I$ denotes the algebra obtained by adjoining an identity operator to $\mathfrak{A}$. The locally convex topology on $\mathfrak{D}$ defined by the seminorms $\{\|\cdot\|_A; A \in \mathfrak{A}_I\}$ is called the induced topology $t_{\mathfrak{A}}$. If $\underline{\mathfrak{D}}(\mathfrak{A}) \equiv \bigcap_{A\in\mathfrak{A}} \mathfrak{D}(\bar{A}) = \mathfrak{D}$, $(\mathfrak{A}, \mathfrak{D})$ is called closed. Let $(\mathfrak{A}, \mathfrak{D})$ be an O_p^*-algebra. An O_p^*-algebra $(\mathfrak{A}, \mathfrak{D})$ satisfying $\bigcap_{A\in\mathfrak{A}} \mathfrak{D}(A^*) = \mathfrak{D}$ is called selfadjoint.

Let $\mathcal{Q}$ be a $*$-algebra which does not necessarily possess an identity. A sesquilinear form ϕ on $\mathcal{Q}$ is called positive invariant if

$$\phi(x, x) \geq 0 \quad \text{and} \quad \phi(ax, y) = \phi(x, a^*y)$$

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for each $a, x, y \in \mathcal{A}$. Let ϕ be a positive invariant sesquilinear form on a $*$-algebra $\mathcal{A}$. Then we obtain the following quartet $(\pi_\phi(\mathcal{A}), \mathfrak{D}_\phi, \lambda_\phi, \mathcal{H}_\phi)$ by the well-known GNS-construction of ϕ, where $\mathfrak{D}_\phi$ is a dense subspace of a Hilbert space $\mathcal{H}_\phi$, $(\pi_\phi(\mathcal{A}), \mathfrak{D}_\phi)$ is a closed O_p^*-algebra and λ_ϕ is a linear map of $\mathcal{A}$ into $\mathfrak{D}_\phi$ (note: $\lambda_\phi(\mathcal{A})$ is dense in $\mathfrak{D}_\phi$ with respect to the induced topology $t_{\pi_\phi(\mathcal{A})}$) satisfying $\lambda_\phi(ax) = \pi_\phi(a)\lambda_\phi(x)$ for each $a, x \in \mathcal{A}$. We put

$$\mathfrak{D}(\pi_\phi^*) = \bigcap_{x \in \mathcal{A}} \mathfrak{D}(\pi_\phi(x)^*), \qquad \pi_\phi^*(x)\xi = \pi_\phi(x^*)^*\xi,$$

for each $x \in \mathcal{A}$ and $\xi \in \mathfrak{D}(\pi_\phi^*)$. Then $(\pi_\phi^*(\mathcal{A}), \mathfrak{D}(\pi_\phi^*))$ is an O_p-algebra. We put

$$\mathfrak{D}(\pi_\phi^{**}) = \bigcap_{x \in \mathcal{A}} \mathfrak{D}(\pi_\phi^*(x)^*), \qquad \pi_\phi^{**}(x)\xi = \pi_\phi^*(x^*)^*\xi,$$

for each $x \in \mathcal{A}$ and $\xi \in \mathfrak{D}(\pi_\phi^{**})$. Then $(\pi_\phi^{**}(\mathcal{A}), \mathfrak{D}(\pi_\phi^{**}))$ is an O_p^*-algebra. The form ϕ is called Riesz if $\mathfrak{D}(\pi_\phi^*) = \mathfrak{D}_\phi$.

3. Strongly cyclic vector representations. In the previous paper **[6]**, we investigated the spatial theory of O_p^*-algebras with a strongly cyclic vector. It will be natural that we consider a generalization, which is defined in the following, of the notion of strongly cyclic vector for an O_p^*-algebra which does not necessarily have an identity.

Let $(\mathfrak{A}, \mathfrak{D})$ be a closed O_p^*-algebra which does not necessarily have an identity and let λ be a linear mapping of $\mathfrak{A}$ into $\mathfrak{D}$. Then λ is called a vector representation if $\lambda(AB) = A\lambda(B)$ holds for each $A, B \in \mathfrak{A}$. In particular, if $\lambda(\mathfrak{A})$ is dense in $\mathfrak{D}$ with respect to the induced topology $t_{\mathfrak{A}}$, then λ is called a strongly cyclic vector representation of $\mathfrak{A}$ into $\mathfrak{D}$. We now consider the following two classes $\mathfrak{X}, \mathfrak{Y}$:

$$\mathfrak{X} = \left\{ (\mathcal{A}, \phi) \;\middle|\; \begin{array}{l} \mathcal{A} \text{ is a } *\text{-algebra and } \phi \text{ is a positive invariant} \\ \text{sesquilinear form on } \mathcal{A} \text{ such that } \pi_\phi \text{ is faithful} \end{array} \right\},$$

$$\mathfrak{Y} = \left\{ (\mathfrak{A}, \mathfrak{D}, \lambda) \;\middle|\; \begin{array}{l} \mathfrak{A} \text{ is a closed } O_p^*\text{-algebra on a pre-Hilbert space } \mathfrak{D} \\ \text{and } \lambda \text{ is a strongly cyclic vector representation} \\ \text{of } \mathfrak{A} \text{ into } \mathfrak{D} \end{array} \right\}.$$

We introduce an equivalence relation on $\mathfrak{Y}$. Let $(\mathfrak{A}, \mathfrak{D}, \lambda)$ and $(\mathfrak{L}, \mathfrak{E}, \mu)$ be elements of $\mathfrak{Y}$. Then $(\mathfrak{A}, \mathfrak{D}, \lambda)$ and $(\mathfrak{L}, \mathfrak{E}, \mu)$ are called equivalent if there exists a unitary transform U of $\mathfrak{D}$ onto $\mathfrak{E}$ such that $U\mathfrak{A}U^+ = \mathfrak{L}$ and $U\lambda(A) = \mu(UAU^+)$ for each $A \in \mathfrak{A}$. In this context, we have the following Proposition.

PROPOSITION. *There exists a map F of $\mathfrak{X}$ into $\mathfrak{Y}$ such that for each $(\mathfrak{A}, \mathfrak{D}, \lambda) \in \mathfrak{Y}$ we can assure the existence of an element $(\mathcal{A}, \phi)$ of $\mathfrak{X}$ whose image by F is equivalent to* $(\mathfrak{A}, \mathfrak{D}, \lambda')$.

PROOF. Let $(\mathcal{A}, \phi)$ be an element of $\mathfrak{X}$. Naturally we have a closed O_p^*-algebra $(\pi_\phi(\mathcal{A}), \mathfrak{D}_\phi)$ by GNS-construction of ϕ. We now define a map λ of $\pi_\phi(\mathcal{A})$ into $\mathfrak{D}_\phi$, putting $\lambda(\pi_\phi(a)) = \lambda_\phi(a)$ for each $a \in \mathcal{A}$ which is well defined since π_ϕ is faithful. Clearly $(\pi_\phi(\mathcal{A}), \mathfrak{D}_\phi, \lambda)$ is an element of $\mathfrak{Y}$. We now define a map F of $\mathfrak{X}$ into $\mathfrak{Y}$, putting $F((\mathcal{A}, \phi)) = (\pi_\phi(\mathcal{A}), \mathfrak{D}_\phi, \lambda)$. In the following we will show the latter part of this Proposition. Let $(\mathfrak{A}, \mathfrak{D}, \lambda)$ be an element of $\mathfrak{Y}$. We now define a positive invariant sesquilinear form ϕ on $\mathfrak{A}$, putting $\phi(x, y) = (\lambda(x) \mid \lambda(y))$ for each $x, y \in \mathfrak{A}$. We consider the closed O_p^*-algebra $(\pi_\phi(\mathfrak{A}), \mathfrak{D}_\phi)$ by GNS-construction of ϕ. Since $\phi(x, y) = (\lambda(x) \mid \lambda(y)) = (\lambda_\phi(x) \mid \lambda_\phi(y))$ for each $x, y \in \mathfrak{A}$, there exists a unitary transform U of $\mathfrak{D}$ onto $\mathfrak{D}_\phi$ such that $U\lambda(x) = \lambda_\phi(x)$ for each $x \in \mathfrak{A}$. Then we have $UaU^+ = \pi_\phi(a)$ for each $a \in \mathfrak{A}$ and $U\mathfrak{A}U^+ = \pi_\phi(\mathfrak{A})$. Thus π_ϕ is faithful and $(\mathfrak{A}, \phi) \in \mathfrak{X}$. When we put $F((\mathfrak{A}, \phi)) = (\pi_\phi(\mathfrak{A}), \mathfrak{D}_\phi, \mu)$, by the definition of F, μ is a map of $\pi_\phi(\mathfrak{A})$ into $\mathfrak{D}_\phi$ defined by $\mu(\pi_\phi(a)) = \lambda_\phi(a)$ for each $a \in \mathfrak{A}$. Thus we have $U\lambda(a) = \lambda_\phi(a) = \mu(\pi_\phi(a)) = \mu(UaU^+)$ for each $a \in \mathfrak{A}$. Therefore $(\mathfrak{A}, \mathfrak{D}, \lambda)$ is equivalent to $(\pi_\phi(\mathfrak{A}), \mathfrak{D}_\phi, \mu) = F((\mathfrak{A}, \phi))$. This completes the proof.

By the above consideration, we see that the investigation of $(\mathfrak{A}, \mathfrak{D}, \lambda) \in \mathfrak{Y}$ is equivalent to that of $(\mathfrak{A}, \phi) \in \mathfrak{X}$.

4. Spatial theory. First of all, we introduce some notation to clarify the object with which we will now deal for spatial theory. Let $\mathcal{A}, \mathfrak{B}$ be $*$-algebras which do not necessarily have an identity. By $I(\mathcal{A}, \mathfrak{B})$ we denote the set of all $*$-isomorphisms of $\mathcal{A}$ onto $\mathfrak{B}$. Let ϕ (resp. ψ) be a positive invariant sesquilinear form on $\mathcal{A}$ (resp. $\mathfrak{B}$). By $I((\mathcal{A}, \phi), (\mathfrak{B}, \psi))$ we denote the set of all $\alpha \in I(\mathcal{A}, \mathfrak{B})$ satisfying $\alpha(\ker \pi_\phi) = \ker \pi_\psi$. Let $(\pi_\phi(\mathcal{A}), \mathfrak{D}_\phi, \lambda_\phi)$ (resp. $(\pi_\psi(\mathfrak{B}), \mathfrak{D}_\psi, \lambda_\psi)$) be the closed O_p^*-algebra by the GNS-construction of ϕ (resp. ψ). Then a $*$-isomorphism $\alpha_{\psi\phi}$ of $\pi_\phi(\mathcal{A})$ onto $\pi_\psi(\mathfrak{B})$ is defined by

$$\alpha_{\psi\phi}\big(\pi_\phi(x)\big) = \pi_\psi(\alpha(x))$$

for each $x \in \mathcal{A}$.

In this paper, we will investigate when $*$-isomorphisms $\alpha_{\psi\phi}$ gained in this manner can be spatially realized. We note that this setting contains the spatial problem between the closed O_p^*-algebras with a strongly cyclic vector representation by the consideration of §3.

THEOREM 4.1. *Let ϕ and ψ be the positive invariant sesquilinear forms on $*$-algebras $\mathcal{A}$ and $\mathfrak{B}$ respectively, and let α be an element of $I((\mathcal{A}, \phi), (\mathfrak{B}, \psi))$.*

(1) *The following statements are equivalent.*

(1.1) *There exists an isometry U of $\mathfrak{D}_\psi$ into $\mathfrak{D}(\pi_\phi^*)$ such that $U\alpha_{\psi\phi}(\pi_\phi(x)) = \pi_\phi^*(x)U$ for each $x \in \mathcal{A}$.*

(1.2) *There exists a linear map μ of $\mathcal{A}$ into $\mathfrak{D}(\pi_\phi^*)$ such that $\mu(ax) = \pi_\phi^*(a)\mu(x)$ and $(\lambda_\psi(\alpha(x)) \mid \lambda_\psi(\alpha(y))) = (\mu(x) \mid \mu(y))$ for each $a, x, y \in \mathcal{A}$.*

(2) *Under the condition of* (1), *the following statements hold.*

(2.1) *There exists an isometry U of $\mathfrak{D}_\psi$ into $\mathfrak{D}(\pi_\phi^*)$ such that $\alpha_{\psi\phi}(\pi_\phi(x)) = U^*\pi_\phi^*(x)U$ for each $x \in \mathcal{A}$. Moreover, $U^*\mathfrak{D}(\pi_\phi^{**}) \subset \mathfrak{D}(\pi_\psi^*)$ and $\pi_\psi^*(\alpha(x))U^*\xi = U^*\pi_\phi^{**}(x)\xi$ for each $x \in \mathcal{A}$ and $\xi \in \mathfrak{D}(\pi_\phi^{**})$.*

(2.2) *If $\pi_\psi^{**} = \pi_\psi^*$ (in particular, if ψ is a Riesz form), we have $\overline{U}U^* \in \pi_\phi(\mathcal{A})'$, where $\pi_\phi(\mathcal{A})'$ denotes the set of all bounded operators A on $\mathcal{H}_\phi$ satisfying $(A\pi_\phi(x)\xi \mid \eta) = (A\xi \mid \pi_\phi(x^*)\eta)$ for each $x \in \mathcal{A}$ and $\xi, \eta \in \mathfrak{D}_\phi$.*

(2.3) *Suppose that $\pi_\psi^* = \pi_\psi$, $\pi_\phi(\mathcal{A})' = \mathbf{C}I$ and $\mu(\mathcal{A}) \subset \mathfrak{D}_\phi$. Then U is a unitary transform of $\mathfrak{D}_\psi$ onto $\mathfrak{D}_\phi$ and ϕ is a Riesz form. Therefore, in this case, we have $\alpha_{\psi\phi}(\pi_\phi(x)) = U^+\pi_\phi(x)U$ for each $x \in \mathcal{A}$, where U^+ denotes the restriction of U^* to $\mathfrak{D}_\phi$.*

PROOF. (1) (1.1) $\Rightarrow$ (1.2). Putting $\mu(x) = U\lambda_\psi(\alpha(x))$ for each $x \in \mathcal{A}$, we easily see that this map μ satisfies all the conditions of (1.2).

(1.2) $\Rightarrow$ (1.1). By the assumption, there exists an isometry U of $\mathfrak{D}_\psi$ into $\mathcal{H}_\phi$ such that $U\lambda_\psi(\alpha(x)) = \mu(x)$ for each $x \in \mathcal{A}$. Let ξ be an arbitrary element of $\mathfrak{D}_\psi$. Then there exists a net $\{y_i\}$ in $\mathcal{A}$ such that $\lambda_\psi(\alpha(y_i))$ converges to ξ in the induced topology $t_{\pi_\psi(\mathcal{B})}$. For each $x, y \in \mathcal{A}$ we have

$$\begin{aligned}\left(\pi_\phi(x^*)\lambda_\phi(y) \mid U\xi\right) &= \lim_i \left(\pi_\phi(x^*)\lambda_\phi(y) \mid U\lambda_\psi(\alpha(y_i))\right) \\ &= \lim_i \left(\pi_\phi(x^*)\lambda_\phi(y) \mid \mu(y_i)\right) = \lim_i \left(\lambda_\phi(y) \mid \pi_\phi^*(x)\mu(y_i)\right) \\ &= \lim_i \left(\lambda_\phi(y) \mid \mu(xy_i)\right) = \lim_i \left(\lambda_\phi(y) \mid U\lambda_\psi(\alpha(xy_i))\right) \\ &= \lim_i \left(\lambda_\phi(y) \mid U\pi_\psi(\alpha(x))\lambda_\psi(\alpha(y_i))\right) = \left(\lambda_\phi(y) \mid U\pi_\psi(\alpha(x))\xi\right).\end{aligned}$$

Hence we have $U\xi \in \mathfrak{D}(\pi_\phi(x^*)^*)$ for each $x \in \mathcal{A}$ and thus we have $U\xi \in \mathfrak{D}(\pi_\phi^*)$ and $\pi_\phi^*(x)U\xi = U\pi_\psi(\alpha(x))\xi$ for each $x \in \mathcal{A}$ and $\xi \in \mathfrak{D}_\psi$. This shows that the isometry U satisfies all the conditions of (1.1).

(2) (2.1) Let U be an isometry of $\mathfrak{D}_\psi$ into $\mathfrak{D}(\pi_\phi^*)$ such that $U\alpha_{\psi\phi}(\pi_\phi(x)) = \pi_\phi^*(x)U$ for each $x \in \mathcal{A}$. It is then clear that $\alpha_{\psi\phi}(\pi_\phi(x)) = U^*\pi_\phi^*(x)U$ for each $x \in \mathcal{A}$. For each $\xi \in \mathfrak{D}(\pi_\phi^{**})$, $\eta \in \mathfrak{D}_\psi$ and $x \in \mathcal{A}$, we have

$$\begin{aligned}\left(\pi_\psi(\alpha(x^*))\eta \mid U^*\xi\right) &= \left(U\pi_\psi(\alpha(x^*))\eta \mid \xi\right) = \left(U\alpha_{\psi\phi}(\pi_\phi(x^*))\eta \mid \xi\right) \\ &= \left(\pi_\phi^*(x^*)U\eta \mid \xi\right) = \left(U\eta \mid \pi_\phi^{**}(x)\xi\right) = \left(\eta \mid U^*\pi_\phi^{**}(x)\xi\right).\end{aligned}$$

Hence we have $U^*\xi \in \mathfrak{D}(\pi_\psi(\alpha(x^*))^*)$ for each $x \in \mathcal{A}$ and thus $U^*\xi \in \mathfrak{D}(\pi_\psi^*)$ and $\pi_\psi^*(\alpha(x))U^*\xi = U^*\pi_\phi^{**}(x)\xi$.

(2.2) We now suppose that $\pi_\psi^{**} = \pi_\psi^*$. Then, since $U^*\lambda_\phi(y) \in \mathfrak{D}(\pi_\psi^*) = \mathfrak{D}(\pi_\psi^{**})$ for each $y \in \mathcal{A}$, it follows that for each $a, x, y \in \mathcal{A}$

$$\begin{aligned}\left(\bar{U}U^*\pi_\phi(a)\lambda_\phi(x) \mid \lambda_\phi(y)\right) &= \left(\pi_\psi^*(\alpha(a))U^*\lambda_\phi(x) \mid U^*\lambda_\phi(y)\right) \\ &= \left(U^*\lambda_\phi(x) \mid \pi_\psi^*(\alpha(a^*))U^*\lambda_\phi(y)\right) \\ &= \left(U^*\lambda_\phi(x) \mid U^*\pi_\phi^{**}(a^*)\lambda_\phi(y)\right) \\ &= \left(\bar{U}U^*\lambda_\phi(x) \mid \pi_\phi(a^*)\lambda_\phi(y)\right).\end{aligned}$$

Hence we have $\bar{U}U^* \in \pi_\phi(\mathcal{A})'$.

(2.3) By the assumption, it is clear that $\bar{U}$ is a unitary transform of $\mathcal{H}_\psi$ onto $\mathcal{H}_\phi$ and $U\mathfrak{D}_\psi \subset \mathfrak{D}_\phi$ (since $U\mathfrak{D}_\psi$ is the $t_{\pi_\phi^*(\mathcal{A})}$-closure of $\mu(\mathcal{A})$). On the other hand, by (2.1) we have $U^+\mathfrak{D}_\phi \subset \mathfrak{D}_\psi$. Thus we have $U\mathfrak{D}_\psi = \mathfrak{D}_\phi$. This shows that U is a unitary transform of $\mathfrak{D}_\psi$ onto $\mathfrak{D}_\phi$. Therefore we have $\alpha_{\psi\phi}(\pi_\phi(x)) = U^+\pi_\phi(x)U$ for each $x \in \mathcal{A}$. It is almost clear that ϕ is a Riesz form. This completes the proof.

Let $\mathcal{A}$ be a $*$-algebra and ϕ, ψ be positive invariant sesquilinear forms on $\mathcal{A}$. Then ψ is called ϕ-bounded if for each $y \in \mathcal{A}$ there exists a positive number γ_y such that $|\psi(x, y)|^2 \leq \gamma_y\phi(x, x)$ for each $x \in \mathcal{A}$. If there exists a positive number γ such that $\psi(x, x) \leq \gamma\phi(x, x)$ for each $x \in \mathcal{A}$, then ψ is called ϕ-dominated. Let $I_b((\mathcal{A}, \phi), (\mathcal{B}, \psi))$ (resp. $I_d((\mathcal{A}, \phi), (\mathcal{B}, \psi))$) be the set of all $\alpha \in I((\mathcal{A}, \phi), (\mathcal{B}, \psi))$

satisfying that ψ_α is ϕ-bounded (resp. ϕ-dominated), where ψ_α is a positive invariant sesquilinear form on $\mathcal{A}$ defined by $\psi_\alpha(x, y) = \psi(\alpha(x), \alpha(y))$ for each $x, y \in \mathcal{A}$.

Before proceeding with our argument, we note the following well-known fact (therefore we omit its proof).

LEMMA 4.2 (FRIEDRICH'S EXTENSION). *Let $\mathfrak{D}$ be a dense linear subspace of a Hilbert space $\mathcal{H}$. Let T_0 be an operator on $\mathcal{H}$ with domain $\mathfrak{D}$. Suppose that $(T_0\xi \mid \xi) \geqslant 0$ for each $\xi \in \mathfrak{D}$. Then there exists a selfadjoint positive operator T such that T is an extension of T_0 and $X\mathfrak{D}(T) \subset \mathfrak{D}(T)$ and $XT\xi = TX\xi$ for each $X \in \mathfrak{B}(T_0)$ and $\xi \in \mathfrak{D}(T)$, where $\mathfrak{D}(T)$ is the domain of T and $\mathfrak{B}(T_0)$ is the set of all bounded operators X on $\mathcal{H}$ satisfying $X\mathfrak{D} \subset \mathfrak{D}$, $X^*\mathfrak{D} \subset \mathfrak{D}$ and $XT_0\xi = T_0X\xi$, $X^*T_0\xi = T_0X^*\xi$ for each $\xi \in \mathfrak{D}$.*

THEOREM 4.3. *Let α be an element of $I_b((\mathcal{A}, \phi), (\mathfrak{B}, \psi))$. Suppose that ϕ is admissible; that is, $\pi_\phi(x)$ is a bounded operator for each $x \in \mathcal{A}$. Then, ψ is also admissible and there exists an isometry U of $\mathcal{H}_\psi$ into $\mathcal{H}_\phi$ such that $\alpha_{\psi\phi}(\pi_\phi(x)) = U^*\pi_\phi(x)U$ for each $x \in \mathcal{A}$.*

PROOF. Putting $K_0\lambda_\phi(x) = \lambda_\psi(\alpha(x))$ for each $x \in \mathcal{A}$, we define an operator K_0 with domain $\lambda_\phi(\mathcal{A})$, which is well defined by the assumption. The inequality $|\psi(\alpha(x), \alpha(y))|^2 \leqslant \gamma_y\phi(x, x)$ means that $|(K_0\lambda_\phi(x) \mid \lambda_\psi(\alpha(y)))| \leqslant \gamma_y^{1/2}\|\lambda_\phi(x)\|$. Hence $\lambda_\psi(\mathfrak{B})$ ($= K_0\lambda_\phi(\mathcal{A})$) is contained in $\mathfrak{D}(K_0^*)$. For each $a, x, y \in \mathcal{A}$, we have

$$\begin{aligned}
\left(K_0^*K_0\pi_\phi(a)\lambda_\phi(x) \mid \lambda_\phi(y)\right) &= \left(\lambda_\psi(\alpha(ax)) \mid \lambda_\psi(\alpha(y))\right) \\
&= \left(\pi_\psi(\alpha(a))\lambda_\psi(\alpha(x)) \mid \lambda_\psi(\alpha(y))\right) \\
&= \left(\lambda_\psi(\alpha(x)) \mid \pi_\psi(\alpha(a^*))\lambda_\psi(\alpha(y))\right) \\
&= \left(K_0\lambda_\phi(x) \mid K_0\lambda_\phi(a^*y)\right) \\
&= \left(K_0^*K_0\lambda_\phi(x) \mid \pi_\phi(a^*)\lambda_\phi(y)\right) \\
&= \left(\pi_\phi(a)K_0^*K_0\lambda_\phi(x) \mid \lambda_\phi(y)\right)
\end{aligned}$$

(since $\pi_\phi(a^*)$ is a bounded operator). Hence we have $K_0^*K_0\pi_\phi(a)\lambda_\phi(x) = \pi_\phi(a)K_0^*K_0\lambda_\phi(x)$ for each $a, x \in \mathcal{A}$. When we use Lemma 4.2 for $K_0^*K_0$, we can easily see the existence of a selfadjoint positive operator H such that H is an extension of $K_0^*K_0$ and affiliated with the von Neumann algebra $\pi_\phi(\mathcal{A})'$. For each $x, y \in \mathcal{A}$, we have

$$\left(H^{1/2}\lambda_\phi(x) \mid H^{1/2}\lambda_\phi(y)\right) = \left(\lambda_\psi(\alpha(x)) \mid \lambda_\psi(\alpha(y))\right).$$

Therefore, when we put $\mu(x) = H^{1/2}\lambda_\phi(x)$ for each $x \in \mathcal{A}$, the map μ satisfies all the conditions of Theorem 4.1(1.2). Thus there exists an isometry U of $\mathfrak{D}_\psi$ into $\mathcal{H}_\phi$ such that $\alpha_{\psi\phi}(\pi_\phi(x)) = U^*\pi_\phi(x)U$ for each $x \in \mathcal{A}$. On the other hand, since

$$\pi_\psi(\alpha(x)) = \alpha_{\psi\phi}\left(\pi_\phi(x)\right) = U^*\pi_\phi(x)U$$

for each $x \in \mathcal{A}$, $\pi_\psi(y)$ is a bounded operator for each $y \in \mathfrak{B}$. Thus $\mathfrak{D}_\psi = \mathcal{H}_\psi$ and ψ is admissible. This completes the proof.

THEOREM 4.4. *Let α be an element of $I_d((\mathcal{A}, \phi), (\mathcal{B}, \psi))$. Suppose that ϕ is a Riesz form. Then there exists an isometry U of $\mathfrak{D}_\psi$ into $\mathfrak{D}_\phi$ such that $\alpha_{\psi\phi}(\pi_\phi(x)) = U^+\pi_\phi(x)U$ for each $x \in \mathcal{A}$.*

PROOF. By the assumption, there exists a bounded operator K of $\mathcal{H}_\phi$ into $\mathcal{H}_\psi$ such that $K\lambda_\phi(x) = \lambda_\psi(\alpha(x))$ for each $x \in \mathcal{A}$. Then we can easily see that $K^*K \in \pi_\phi(\mathcal{A})'$. Since $\pi_\phi(\mathcal{A})'$ is a von Neumann algebra, $(K^*K)^{1/2}$ is an element of $\pi_\phi(\mathcal{A})'$. Now we put $\mu(x) = (K^*K)^{1/2}\lambda_\phi(x)$. Then it is almost clear that μ satisfies all the conditions of Theorem 4.1(1.2). Hence there exists an isometry U of $\mathfrak{D}_\psi$ into $\mathfrak{D}_\phi$ such that $\alpha_{\psi\phi}(\pi_\phi(x)) = U^+\pi_\phi(x)U$ for each $x \in \mathcal{A}$. This completes the proof.

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