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L^p-BEHAVIOR OF POWER SERIES WITH POSITIVE COEFFICIENTS AND HARDY SPACES

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ABSTRACT. For the power series $f(x) = \sum_1^\infty a_n x^n$ with $a_n \geq 0$, certain weighted L^p-norms of f on $[0,1]$ are estimated from above and below in terms of the coefficients a_n. Some consequences of this are obtained. For example, some known results concerning Hardy spaces may be extended to a wider class of spaces.

0. Introduction. Our main tool here is the following theorem.

THEOREM 1. *If $\alpha > 0$, $p > 0$, $n \geq 0$, $n \in N$, $a_n \geq 0$, $I_n = \{k\colon 2^n \leq k < 2^{n+1}, k \in N\}$, $t_n = \sum_{k \in I_n} a_k$ and $f(x) = \sum_1^\infty a_n x^n$, then there is a constant K which depends only on p and α such that*

$$\frac{1}{K}\sum_0^\infty 2^{-n\alpha} t_n^p \leq \int_0^1 (1-x)^{\alpha-1} f(x)^p\,dx \leq K \sum_0^\infty 2^{-n\alpha} t_n^p.$$

Our proof is based on Jensen's inequality and other elementary inequalities. The proof shows that the theorem is still valid if t_n is replaced by $S_2 n$, where $S_n = \sum_1^n a_k$. Hence, by the Cauchy condensation test which holds for series whose terms are quasimonotone, we obtain

COROLLARY 1. *Under the condition of Theorem 1, there is a constant K which depends only on p and α such that*

$$\frac{1}{K}\sum_1^\infty n^{-(\alpha+1)} S_n^p \leq \int_0^1 (1-x)^{\alpha-1} f(x)^p\,dx \leq K \sum_1^\infty n^{-(\alpha+1)} S_n^p.$$

For $\alpha = 1$ this result is due to Boas and Askey [**1**; **2**, Theorem 2].

In addition, Theorem 1 can be used to provide easy proofs of some results of Hardy-Littlewood [**5**] concerning power series with positive coefficients (see §2).

We proceed to elucidate the connection between our result and some known results on Hardy spaces. From now on we assume, without loss of generality and with great gain in convenience that all regular functions under discussion vanish at the origin. In particular, the H^p spaces used below are thus subsets of the usual H^p spaces (cf. Duren [**3**]).

DEFINITION 1. If $f(z) = \sum_1^\infty a_n z^n$ is regular on the unit disc, $p > 0$ and $\alpha > 0$, we write $f \in D_\alpha^p$, whenever

$$\|f\|_{D_\alpha^p}^p := \sum_0^\infty 2^{-n\alpha} r_n^p(f) < +\infty \tag{0.1}$$

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where $r_n(f) = (2^n \sum_{k\in I_n} |a_k|^2)^{1/2}$. D^∞_α can be defined analogously. We also write D^p instead of D^p_1.

For $p \geq 1$, D^p_α is a Banach space under the norm defined by (0.1); and for $0 < p < 1$ D^p_α (with the obvious translation metric) is a complete F-space. It is easy to check that, if $1 \leq p < \infty$, then D^q_α is the Banach dual of D^p_α, where $1/p + 1/q = 1$; and if $0 < p < 1$, then the conjugate space may be identified with the space of regular functions $g(z) = \sum_1^\infty b_k z^k$ in the unit disc for which

$$\sum_{k\in I_n} k|b_k|^2 = O(2^{2n(\alpha-\alpha/p)}) \qquad (n \to \infty).$$

The dual pairing in all cases is given by formula $(f, g) = \sum_1^\infty k^{1-\alpha} a_k b_k$. Holland and Twomey in [**9**] defined l^ξ_p spaces. Theorem 1 and Corollary 1 show that D^p and l^ξ_p may be identified. Holland and Twomey [**8, 9**] show $H^p \subset D^p$ for $0 < p \leq 2$, $D^p \subset H^p$ for $p \geq 2$ and that both inclusions are proper for $p \neq 2$.

The following Theorem 2 generalizes this result. Our proof of Theorem 2 resembles that of Holland and Twomey. Both proofs are based on the Hardy-Stein identity (see §2). Their proof also uses the above-mentioned Askey-Boas result and the Littlewood inequality [**4, 12**]; Theorem 1 and Proposition 2, respectively, play analogous roles in our proof. Note that Proposition 2 is a generalization of the Littlewood inequality for $0 < p \leq 2$ (aside from the value of the constant), where the H^p-norm is replaced by the D^p-norm. Also note this proposition is an immediate consequence of Theorem 1.

Now it is natural to ask whether there are any theorems on H^p spaces which can be extended to the wider class D^p for $0 < p < 2$. The answer is yes: in §2, among other things, we improve a result of Hardy-Littlewood on fractional integrals.

1. Bergman class and the area function.

Definitions of classes A^p_α and H^p_α. Let $f(z) = \sum_1^\infty a_n z^n$ be regular on the unit disc. We define the functions

$$P(r) = P(r, f) = \sum_1^\infty |a_n| r^n,$$

$$A(r) = A(r, f) = \iint_{|z|\leq r} |f'(z)|^2\, dx\, dy = \pi \sum_1^\infty n|a_n|^2 r^{2n},$$

where $0 \leq r < 1$. The function $A(r)$ is called the area function. In fact, $A(r)$ is the area of the image of $\{z\colon |z| \leq z\}$ under f, counting multiplicity.

We shall write $f \in A^p_\alpha$, $p > 0$, $\alpha > 0$, whenever

$$\|f\|^p_{A^p_\alpha} := \int_0^1 A(r, f)^{p/2} (1 - r)^{\alpha - 1}\, dr < \infty. \tag{1.1}$$

The Bergman class B^p_α $(p > 0, \alpha > 1)$ consists of all functions f regular on the unit disc for which

$$\int_0^1 I_p(r, f)(1 - r)^{\alpha-2}\, dr < +\infty, \tag{1.2}$$

where

$$I_p(r, f) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta, \qquad 0 \leq r < 1.$$

We shall use the Hardy-Stein identity [7, p. 42]

$$r\frac{d}{dr}I_p(r,f)=\frac{p^2}{2\pi}E_p(r,f)\qquad (0<r<1, p>0),$$

where

$$E_p(r)=E_p(r,f)=\iint_{|z|<r}|f'(z)|^2|f(z)|^{p-2}\,dx\,dy.$$

Also, we shall use the notation $f\in H^p_\alpha$, $p>0$, $\alpha>0$, whenever

$$\|f\|^p_{H^p_\alpha}=\int_0^1 E_p(r,f)(1-r)^{\alpha-1}\,dr<+\infty. \tag{1.3}$$

Integration by parts of (1.2) shows that $H^p_\alpha=B^p_\alpha$ for $\alpha>1$, $p>0$.

Thus we can extend the definition of B^p_α for $\alpha>0$ by (1.3). The symbol H^p_α is motivated by the fact that $H^p_1=H^p$.

An extension of the result of Holland and Twomey. The following are immediate consequences of Theorem 1.

PROPOSITION 1. *The spaces D^p_α and A^p_α $(p>0,\alpha>0)$ are the same and the "norms" defined by $(0\cdot 1)$ and (1.1) are equivalent.*

PROPOSITION 2. *Let $f\in D^p_\alpha$, $p>0$, $\alpha>0$. Then there is a constant K which depends only on p and α such that*

$$\int_0^1 P(r)^p(1-r)^{\alpha-1}\,dt\le K\|f\|^p_{D^p_\alpha},$$

for all $f\in D^p_\alpha$.

This proposition is a generalization of Hardy-Littlewood's result [6, Theorem 15, p. 206].

PROOF. Apply first Theorem 1 and then the Cauchy-Schwarz inequality $\sum_{k\in I_n}|a_k|\le r_n(f)$. Now, we are ready to state the main result of this section.

THEOREM 2. *Let $\alpha>0$. Then*

$$H^p_\alpha\subset D^p_\alpha\quad \text{if } 0<p\le 2, \tag{1.4}$$

$$D^p_\alpha\subset H^p_\alpha\quad \text{if } p\ge 2. \tag{1.5}$$

PROOF. Suppose first that $0<p\le 2$ and $f\in H^p_\alpha$. Let

$$\varphi(r)=P(r,f)^p(1-r)^{\alpha-1}.$$

Combining the inequality $E_p(r,f)\ge P(r)^{p-2}A(r)$ with Jensen's inequality (for the concave action $t^{p/2}$), we obtain

$$\begin{aligned}\|f\|^{p^2/2}_{H^p_\alpha}&\ge\left\{\int_0^1 P(r,f)^{-2}A(r,f)\varphi(r)\,dr\right\}^{p/2}\\ &\ge\left\{\int_0^1\varphi(r)\,dr\right\}^{p/2-1}\|f\|^p_{A^p_\alpha}\end{aligned}$$

Now, (1.4) follows from Propositions 1 and 2.

Suppose now that $p \geq 2$ and $f \in A^p_\alpha$. Combining the inequality $E_p(r,f) \leq P(r,f)^{p-2}A(r,f)$ and Hölder's inequality, we have

$$\|f\|^p_{H^p_\alpha} \leq \|f\|^2_{A^p_\alpha}\left\{\int_0^1 P(r,f)^p(1-r)^{\alpha-1}\,dr\right\}^{(p-2)/p}.$$

Application of Propositions 1 and 2 yields (1.5).

The examples $f(z) = \sum a_n z^{2^n}$, $g(z) = \sum b_n z^{2^n}$, where $2^{-n(\alpha-p/2)}|a_n|^p = n^{-2}$ and $2^{-n(\alpha-p/2)}|b_n|^p = 1$ shows respectively that the inclusions (1.4) and (1.5) are proper if $p \neq 2$.

We shall show that $f \notin H^p_\alpha$. The rest of the proof is similar. Suppose first that $\alpha > p/2$. A direct calculation based on the Cauchy condensation test shows that

$$P(r,f) \leq K(1-r)^{-\beta},$$
$$A(r,f) \geq \frac{1}{K}(1-r)^{-\gamma}, \quad 0 \leq r < 1,$$

where $\beta = \alpha/p - 1/2$, $\gamma = 2\beta + p/2$ and K is a positive constant. Now, the desired result follows from the inquality $E_p(r,f) \geq P(r)^{p-2}A(r)$.

For $\alpha \leq p/2$ the proof is simpler, because the function $P(r,f)$, $0 \leq r < 1$, is bounded.

COROLLARY 2. *If $0 < q < 1$, then $B^q \subset D^1_{1/q}$, where B^q is the Banach envelope of H^q.*

2. Some consequences of Theorem 1.

Fractional derivatives and the area function. For a function f regular in the unit disc we denote by $f^{[\beta]}$ ($f_{[\beta]}$), $\beta \leq 0$, the fractional derivative (integral) of order β. Recall that

$$f^{[\beta]}(z) = \sum_0^\infty \frac{\Gamma(n+1+\beta)}{n!}a_n z^n,$$
$$f_{[\beta]} = \sum_0^\infty \frac{n!}{\Gamma(n+1+\beta)}a_n z^n.$$

In these notations we have

THEOREM 3. *Let $p > q > 0$, $\beta > 0$, $\alpha > 0$ and $\beta/\alpha = 1/q - 1/p$. The following assertions hold:*

(a) *If $f \in D^q_\alpha$, then $f_{[\beta]} \in D^p_\alpha$. There is a function f such that $f \in D^q_\alpha$ but $f_{[\beta]} \notin D^\mu_\alpha$ for all $\mu > p$.*

(b) *If $f \in D^p_\alpha$, then $f^{[\beta]} \in D^\mu_\alpha$ for $\mu < q$. There is a function f such that $f \in D^p_\alpha$ and $f^{[\beta]} \notin D^q_\alpha$.*

In the case $p > 2 > q$, $\alpha = 1$, the assertion (a) improves the corresponding one for H^p spaces. The statement (b) has no analog in H^p because f' need not belong to the Nevanlinna class for f bounded in the unit disc.

PROOF. From Definition 1 and $\Gamma(n+1+\beta)/n! \sim n^\beta$, $n \to \infty$, it follows that $f_{[\beta]} \in D^p_\alpha$ ($f \in D^p_\alpha$) if and only if $f \in D^p_{\alpha+\beta p}$ ($f^{[\beta]} \in D^p_{\alpha+\beta p}$). Now, Theorem 3 can be derived from the following proposition.

PROPOSITION 3. *Let $p > 0$, $\alpha > 0$ and $\mu > \lambda \geq 1$. Then $A^{\mu p}_{\lambda\alpha} \subset A^p_\alpha \subset A^{\lambda p}_{\lambda\alpha}$. For fixed p, α and λ there is a function f such that $f \in A^p_\alpha$ and $f \notin A^{\mu p}_{\lambda\alpha}$ for any $\mu > \lambda$. If $\lambda > 1$, then $A^p_\alpha \neq A^{\lambda p}_{\lambda\alpha}$.*

PROOF. By Hölder's inequality, we obtain

$$||f||_{A^p_\alpha} \le ||f||_{A^{\mu p}_{\lambda\alpha}} \left\{ \frac{\mu - 1}{\alpha(\mu - \lambda)} \right\}^{(\mu-1)/\mu p}.$$

This proves the left inclusion. The example $f(z) = \sum_1^\infty a_n z^{2^n}$, where $2^{-n(\alpha - p/2)}|a_n|^p = n^{-2}$ shows that this inclusion is proper. The rest of the proof is simple.

Further consequences of Theorem 1.

COROLLARY 3. *Let $f' \in D^p$ for some $p > 1$. Then $\sum_0^\infty |a_n| < \infty$ and $f(e^{i\theta}) \in \Lambda_{1-1/p}$, where Λ_α $(0 < \alpha \le 1)$ denotes the class of all complex-valued 2π-periodic functions on $(-\infty, +\infty)$ which satisfy the Lipschitz condition of order α.*

In the case $1 < p < 2$ this result improves the corresponding one for H^p because $H^p \subset D^p$ (for $p < 2$).

PROOF. Let $f' \in D^p$. Then, by Proposition 2, $\int_0^1 P(r, f')^p \, dr < +\infty$ and, therefore,

$$P(r, f') = O(1 - r)^{-1/p}, \qquad r \to 1_-.$$

Hence,

$$\sum_0^\infty |a_n| < \infty \quad \text{and} \quad f'(z) = O(1-r)^{-1/p}, \qquad r \to 1_-,$$

and, by the Hardy-Littlewood theorem [**3**, Theorem 5.1] $f \in \Lambda_{1-1/p}$.

F. Holland and J. B. Twomey [**8**] have proved that there is a function f such that $f \in H^p$ for each p, $0 < p < \infty$, while $f \notin D^p$ for $p > 2$. Here, we observe that $h(z) = \sum_1^\infty n^{-2} z^{2^n} \in H^\infty$, but $h \notin D^p$ for $p > 2$. This follows from Theorem 1.

On the other hand it is clear that $H^\infty \subset D^p$ if $0 < p \le 2$, because $D^2 = H^2$. The following stronger results hold.

THEOREM 4. *Let $0 < p \le 2$, $g \in H^\infty$ and $f \in D^p$. Then $f(z)g(z) \in D^p$.*

PROOF. In the case $p = 2$ the assertion is trivial. Suppose that $0 < p < 2$. From the condition $g \in H^\infty$ it follows that $|g'(z)| \le K(1 - |z|)^{-1}$ $(|z| < 1)$, for some constant K. Hence

$$F(r) := \iint_{|z| \le r} |f(z)g'(z)|^2 \le K_1 (1-r)^{-1} \sum_1^\infty |a_k|^2 r^{2k}, \qquad 0 \le r < 1,$$

where K_1 is a positive constant. From the condition $f \in D^p$ and Theorem 1 it follows $\int_0^1 F(r)^{p/2} \, dr < \infty$. The statement may easily be derived from Proposition 1 and the inequality

$$\iint_{|z| \le r} |f'(z)|^2 |g(z)|^2 \, dx \, dy \le ||g||_\infty^2 A(r, f).$$

THEORERM 5. *Let $f(x) = \sum_1^\infty a_n x^n$, $a_n \ge 0$,*

(i) *If $q > 0$, $r \ge p > 1$, then*

$$\int_0^1 (1-x)^{r/q - 1} f^r(x) \, dx \le K_1 \left(\sum_1^\infty n^{-(p+q-pq)/q} a_n^p \right)^{r/p}.$$

(ii) *If $q > 0$, $0 < r \le p < 1$, then*

$$\sum_1^\infty n^{-(p+q-pq)/q} a_n^p \le K_2 \left(\int_0^1 (1-x)^{r/q-1} f^r(x) \, dx \right)^{p/r},$$

where K_1 and K_2 depend on p, q and r only.

These results first appear in Hardy-Littlewood [**5**, Theorem 3 and 11].

PROOF. Let us first consider (i). By Theorem 1 and Jensen's inequality,

$$\int_0^1 (1-x)^{r/q-1} f^r(x)\,dx \le K \sum_0^\infty 2^{nr(1-1/q)} \left(\sum_{k\in I_n} 2^{-n} a_k \right)^r \le K \sum_0^\infty 2^{nr(1-1/q)} \left(\sum_{k\in I_n} 2^{-n} a_k^p \right)^{r/p}$$

where K depends on p, q and r only. Now we use elementary inequalities and a Cauchy condensation type argument to obtain (i). Part (ii) can be derived in a similar way.

3. Proof of Theorem 1. Theorem 1 follows immediately from a more general fact which, because of its independent interest, we state as a theorem. We need the following notations: $\Phi \in \Delta(p,q)$ $(p \ge q > 0)$ if Φ is a nonnegative real function defined on $[0,+\infty)$, $\Phi(0)=0$, $\Phi(t)/t^p$ is nonincreasing and $\Phi(t)/t^q$ is nondecreasing. We write $\Phi\in\Delta$ if $\Phi\in\Delta(p,q)$ for some p and q. For example, $t^p \in \Delta(p,p)$ and $t^p\log(1+t) \in \Delta(p+1,p)$ if $p>0$.

THEOREM 6. *Let (a_k), $k \ge 1$, be a sequence of nonnegative real numbers, $\Phi\in\Delta$ and $\alpha > 0$. Then $\int_0^1 \Phi(\sum_{k=1}^\infty a_k r^k)(1-r)^{\alpha-1}\,dr < \infty$ if and only if $\sum_0^\infty 2^{-2n\alpha}\Phi(\sum_{I_n} a_k) < \infty$.*

For the proof of this theorem we need some elementary facts.

LEMMA 1. *Let $\Phi \in \Delta(p,q)$ $(p\ge q>0)$ and $t_n \ge 0$ $(n=1,2,\ldots)$. Then*

$$\theta^p\Phi(t) \le \Phi(\theta t) \le \theta^q \Phi(t) \quad \text{for } 0\le\theta\le 1,\ t\ge 0, \tag{3.1}$$

$$\Phi\left(\sum_1^\infty t_n\right) \le \left(\sum_1^\infty \Phi(t_n)^{1/p}\right)^p, \tag{3.2}$$

$$\Phi\left(\sum_1^\infty t_n\right) \le \sum_1^\infty \Phi(t_n), \qquad 0<p\le 1. \tag{3.3}$$

PROOF. (3.1) follows immediately from definitions. Observe that Φ is continuous (by (3.1)) and, therefore, we may suppose that $t_n = 0$ for $n\ge 3$. Since $\Phi(t)^{1/p}/t$ is nonincreasing, $\Phi(t)^{1/p}$ is subadditive and hence (3.2). From (3.2) and elementary inequalities we derive (3.3).

LEMMA 2. *Let $0<\gamma\le 1$ and $\eta(r) = \sum_0^\infty 2^{n\gamma} r^{2^n}$ $(0\le r<1)$. Then $\eta(r) \le 2\Gamma(\gamma)|\log r|^{-\gamma}$.*

PROOF. We have

$$2\Gamma(\gamma)|\log r|^{-\gamma} = 2^{1-\gamma}\int_0^{+\infty} t^{\gamma-1} r^{t/2}\,dt \ge 2^{1-\gamma}\sum_1^\infty k^{\gamma-1} r^{k/2} \ge 2^{1-\gamma}\sum_0^\infty 2^n 2^{(n+1)(\gamma-1)} r^{2^n} = \eta(r).$$

Proof of Theorem 6. ($\Rightarrow$) Let $t_n = \sum_{I_n} a_k$ and $r_n = 1-2^{-n}$ $(n=0,1,\dots)$. Then

$$\begin{aligned}\int_0^1 \Phi\left(\sum_1^\infty a_n r^n\right)(1-r)^{\alpha-1}\,dr &\geq \int_{1/2}^1 \Phi\left(\sum_0^\infty t_k r^{2^{k+1}-1}\right)(1-r)^{\alpha-1}\,dr \\ &\geq \sum_0^\infty \Phi\left(\sum_{k=0}^n t_k r_{n+1}^{2^{k+1}-1}\right)\int_{r_{n+1}}^{r_{n+2}}(1-r)^{\alpha-1}\,dr \\ &\geq \sum_0^\infty \Phi\left(e^{-1}\sum_{k=0}^n t_k\right)2^{-(n+2)\alpha}\log 2 \\ &\geq K_1\sum_0^\infty \Phi\left(\sum_{k=0}^n t_k\right)2^{-n\alpha},\end{aligned}$$

where $K_1 = e^{-p}2^{-2\alpha}\log 2$. Hence,

$$\int_0^1 \Phi\left(\sum_1^\infty a_n r^n\right)(1-r)^{\alpha-1}\,dr \geq K_1\sum_0^\infty 2^{-n\alpha}\Phi(S_2 n).$$

($\Leftarrow$) Let us first suppose that $p>1$. Because of the inequality (3.2) we can take $\Phi(t)=t^p$.

Let $\gamma = \min\{1,\alpha/p\}$ and $\eta(r) = \sum_0^\infty 2^{n\gamma}r^{2^n}$, $0\leq r<1$. Then, by Jensen's inequality, we have

$$\left(\sum_1^\infty a_k r^k\right)^p \leq \left(\sum_0^\infty t_n r^{2^n}\right)^p \leq \eta(r)^{p-1}\sum_0^\infty 2^{n\gamma}r^{2^n}2^{-n\gamma p}|t_n|^p.$$

Now, the desired result follows from Lemma 2, the inequality $r(1-r)^{\alpha-1}\leq|\log r|^{\alpha-1}$, $0\leq r<1$, and the equality $\int_0^1|\log r|^{\gamma-1}r^{2^n-1}\,dr = \Gamma(\gamma)2^{-n\gamma}$. A direct calculation shows that

$$\int_0^1\left(\sum_1^\infty a_k r^k\right)^p(1-r)^{\alpha-1}\,dr \leq K_2\sum_0^\infty 2^{-n\alpha}\left(\sum_{I_n} a_k\right)^p,$$

where (in the case $p\geq\alpha$) $K_2 = 2^{p-1}\Gamma(\gamma)^p$.

For $p=1$, the proof is similar, but simpler, because we need not introduce the function $\eta(r)$. In the case $0<p<1$ the relation (3.3) is used, and the proof resembles that for $p=1$.

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