ON THE ADAMS CONJECTURE

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ABSTRACT. We give a short elementary and self-contained proof to the
Adams conjecture. We then present three equivalent formulations of the
conjecture.

0. Introduction. The purpose of this note is to give a short elementary and
self-contained proof to the Adams conjecture [10, 11], in particular assuming only
Dold's Theorem mod k. Our argument is a variation, using the elementary notion
of degree functions, on the Becker-Gottlieb proof [6] which has the advantage of
making no appeal to infinite loop space structures on SG and BSG.

We define a nondegenerate form q called the degree function on the group
KF°r(B) of stable fibre homotopy classes of orientable spherical Hurewicz fibrings
over B. The crucial point in our argument is Proposition 1.5.1 which states that
if p: E — B is a fibre bundle with fibre F then p': KF°r(B) — KF°r(E) satisfies
q(u)x(F)q(p'(u)), u G KF°r(B) where x(F) is the Euler-Poincaré characteristic of
F. This enables us to extend the proof of the Adams conjecture from N(T) to
SO(2n)-bundles where N(T) is the normalizer of the maximal torus T in SO(2n).
We also eliminate the inessential use in [6] of the transfer for finite coverings.

In §2 we present three alternative formulations of the Adams conjecture and
prove their equivalence. Of these the third one gives more information than the
usual statement of the conjecture.

1.1. The fibrewise map f_k(ξ): E_ξ — E_yk(ξ), for N(T)-bundles. Let V = R^{2n}
be the N(T)-module obtained by restricting the usual SO(2n)-action to N(T).
Decompose V = V_1 © · · · © V_n where each V_i is isomorphic to R^2 and is spanned
by basis elements e_{2i-1}, e_{2i}. The permutation group S_n acts on V by permuting
the factors V_i. Let u_i G O(2n) permute the basis elements e_{2i-1} and e_{2i} and leave
V_j for j ≠ i pointwise fixed. Products of an even number of u_i’s form a subgroup
U_n of SO(2n) of order 2^{n-1}. Let W_n = S_n x U_n be their twisted cross product
which is a subgroup of SO(2n) of order 2^{n-1}n!. Let T = T^n be the maximal torus
in SO(2n) and N(T) be its normalizer. Then (i) N(T) is generated by W_n and T
multiplicatively and (ii) The Weyl group, N(T)/T = W_n.

1.1.1. LEMMA. Let ϕ_k: T — T be the homomorphism which raises each coordinate
to its kth-power, i.e. ϕ_k([z_1, · · · , z_n]) = [z_1^k, · · · , z_n^k]. Then ϕ_k extends to a
homomorphism, ̃ϕ_k: N(T) — N(T).

PROOF. Define ̃ϕ_k: N(T) — N(T) by ̃ϕ_k(ut) = wϕ_k(t), ∀w G W_n, t G T. It is
easily verified that ̃ϕ_k is well defined, is a homomorphism and extends ϕ_k. Q.E.D.

Let ψ_k denote the Adams operation as defined in [1].

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1.1.2. Corollary. As an $N(T)$-module, $\psi^k(V) = \tilde{\phi}_k^*(V)$. In particular, $\psi^k$ maps an $N(T)$-module to a pure $N(T)$-module and not a virtual one.

**Proof.** It follows from the definition of $\psi^k(V)$ and Lemma 1.1.1 above.

1.1.3. Corollary. Let $P$ be a principal $N(T)$-bundle over a topological space $B$ and $E_\xi = P \times_{N(T)} V \to B$ define a vector-bundle $\xi$ on $B$. Then $\psi^k(\xi)$ is a pure vector-bundle and its total space, $E_{\psi^k(\xi)} = P \times_{N(T)} \tilde{\phi}_k^*(V) = P \times_{\tilde{\phi}_k} V$.

**Proof.** It follows from the definition of $\psi^k(\xi)$ and Corollary 1.1.2 above.

1.1.4. Lemma. Let $P$ be a principal $N(T)$-bundle over a topological space $B$, and $E_\xi = P \times_{N(T)} V \to B$ define a vector-bundle $\xi$ on $B$. Let $S(\xi)$ be the associated sphere bundle. Then there exists a fibrewise map, $f_k(\xi): E_{S(\xi)} \to E_{S(\psi^k(\xi))}$ of degree $k^n$ at each fibre.

**Proof.** Let $p_k: S^1 \to S^1$ be the $k$th-power map defined by $p_k(z) = z^k$, $\forall z \in S^1$. Define $f_k: S(V) = S(V_1) \ast \cdots \ast S(V_n) \to S(V_1) \ast \cdots \ast S(V_n) = (\tilde{\phi}_k^*(V))$ be the $n$-fold join, $f_k = p_k \ast \cdots \ast p_k$. Then (i) $f_k$ is equivariant under the $N(T)$-actions on $S(V)$ and $S(\tilde{\phi}_k^*(V))$ and (ii) $f_k$ defines a bundle map, $f_k(\xi): E_{S(\xi)} = P \times_{N(T)} S(V) \to P \times_{N(T)} (\tilde{\phi}_k^*(V)) = E_{S(\psi^k(\xi))}$, of degree $k^n$ at each fibre. Q.E.D.

1.1.5. Remark. If $\xi$ has a spin reduction, i.e. if its structure group is reducible to the normalizer $N(T^{4n})$ of the maximal torus $T^{4n}$ in Spin$(8n)$ and $f_k(\xi)$: $B^\xi \to B^{\psi^k(\xi)}$, the map on the Thom complexes induced by $f_k(\xi)$ and $U_\xi \in K^R(B^\xi)$ and $U_{\psi^k(\xi)} \in K^R(B^{\psi^k(\xi)})$ the respective $K^R$-theory Thom classes then $f_k(\xi)(U_{\psi^k(\xi)}) = \rho_R^k(U_{\xi})$ where $\rho_R^k$ is the characteristic class associated to the Adams operation $\psi_R^k$ via the $K^R$-theory Thom isomorphism as defined in [7] and [3].

The proof of this is quite straightforward and is left to the reader since it is not needed in our proof of the Adams conjecture.

1.2. Spherical Hurewicz fibrings. A fibration which is locally fibre homotopy equivalent to the product of the base space and a sphere is called a spherical Hurewicz fibring. The direct sum of two spherical Hurewicz fibrings is defined by taking joins at each fibre. The Thom complex $B^\xi$ of a spherical Hurewicz fibring $\xi$ over $B$ is defined by $B^\xi = E_{S(\xi)}$ where $s: B \to E_{S(\xi)}$ is the section at infinity. If $\xi$ is a vector-bundle and $S(\xi)$ the associated sphere bundle then the Thom complex $B^{S(\xi)}$ of $S(\xi)$ coincides with the Thom complex $B^\xi$ of $\xi$. We shall adopt the notation $B^{S(\xi)}$ for this space.

1.3. The transfer map $t_\xi(\xi)$. In this section we consider a slight modification of the transfer map of [6]. We thus rely heavily on §3 of [6] for notation and results. Let $G$ be a compact Lie group, $P$ a principal $G$-bundle over a topological space $B$, $F$ a compact smooth $G$-manifold and $E = P \times_G F \to B$, the associated fibre bundle over $B$ with fibre $F$. We imbed $F$ equivariantly in a representation module $V$ of $G$. We then have as in 3.5 of [6] an equivariant map $\gamma: S^V \to F^+ \wedge S^V$ and as in 3.6 of [6] an ex-map $\gamma' = 1 \times_G \gamma: P \times_G S^V \to P \times_G (F^+ \wedge S^V)$. Let $E_\eta = P \times_G V \to B$ be the associated vector-bundle to $P$ with fibre $V$ and $\xi$ a complementary vector-bundle to $\eta$ with fibre $W$ and trivialization $\phi: E_{\eta \oplus W} \to B \times R^2$. Let $\xi$ be an arbitrary $S^{n-1}$-fibration over $B$. Let $\bar{\xi} = S(\xi \oplus 1)$ and $\bar{\xi} = \xi \oplus 1$.

$$\gamma' \wedge_B 1 \wedge_B 1: (P \times_G S^V) \wedge_B E_{\bar{\xi}} \wedge_B E_{\bar{\xi}} \to (P \times_G (F^+ \wedge S^V)) \wedge_B E_{\bar{\xi}} \wedge_B E_{\bar{\xi}}.$$
If we identify $B$ to a point on each side, the resulting quotient space on the left is $B^{S(\eta \oplus \xi)} \oplus S$ whereas the one on the right is $E^{p} S(\eta \oplus \xi) \oplus E^{p} (\xi)$. Let $\sigma: B^{S(\eta \oplus \xi) \oplus \xi} \rightarrow E^{p} S(\eta \oplus \xi) \oplus E^{p} (\xi)$ be the induced map. We define $t_{p}(\xi)$ as the composite

$$t_{p}(\xi): S^{*} B^{\xi} \overset{p^{-1}}{\rightarrow} B^{S(\eta \oplus \xi) \oplus \xi} \overset{\sigma}{\rightarrow} E^{p} S(\eta \oplus \xi) \oplus E^{p} (\xi) \overset{P(\xi)}{\rightarrow} S^{*} E^{p} (\xi).$$

1.3.1. LEMMA. Let $U_{s \oplus \xi} \in H^{n+s}(S^{*} B\xi)$ and $U_{s \oplus p}(\xi) \in H^{n+s}(S^{*} E^{p}(\xi))$ be the cohomology Thom classes. Then $t_{p}(\xi)(U_{s \oplus p}(\xi)) = \chi(F) U_{s \oplus \xi}$.

**Proof.** We have a commutative diagram

$$
\begin{array}{ccc}
(S^{n} \wedge S^{W}) \wedge S^{V} & \overset{1 \wedge \gamma}{\rightarrow} & (S^{n} \wedge S^{W}) \wedge (F^{+} \wedge S^{V}) \\
\downarrow j & & \downarrow k \\
S^{*} B^{\xi} & \overset{t_{p}(\xi)}{\rightarrow} & S^{*} E^{p}(\xi) \\
\end{array}
$$

where $j$ and $k$ are inclusions over some point of $B$ and $\pi: F^{+} \wedge S^{V} \rightarrow S^{V}$ is the projection. Hence if $t \in H^{*}(S^{n} \wedge S^{W})$ and $\iota_{V} \in H^{*}(S^{V})$ are the generators in positive dimension then $j^{*}(U_{s \oplus \xi}) = t \otimes \iota_{V}$ and $k^{*}(U_{s \oplus p}(\xi)) = t \otimes \iota_{V}$. By Theorem 2.6 of [8] we have $\gamma^{*} \circ \pi^{*}(\iota_{V}) = \chi(F) \iota_{V}$ and hence $(1 \wedge \gamma)^{*} \circ (1 \wedge \pi)^{*}(\iota \otimes \iota_{V}) = \chi(F)(\iota \otimes \iota_{V})$. Thus

$$
t_{p}(\xi) U_{s \oplus \xi} = m \iota_{V}(\xi) U_{s \oplus \xi} \quad \text{for some integer } m.
$$

Hence $m = \chi(F)$ and $t_{p}(\xi) U_{s \oplus \xi} = m \iota_{V}(\xi) U_{s \oplus \xi}$.

**Q.E.D.**

1.4. The degree function $q$ on $K\overline{F}^{or}(B)$. Let $B$ be a finite dimensional CW-complex and $K\overline{F}(B)$ the group of stable fibre homotopy classes of spherical Hurewicz fibrings over $B$ and $K\overline{F}^{or}(B)$ its subgroup generated by orientable fibrings. The $J$-homomorphism, $J: K\overline{F}(B) \rightarrow K\overline{F}^{or}(B)$, maps a vector-bundle into its associated sphere-bundle. The image of $J$ is defined to be the group $J(B)$.

Let $\xi$ and $\eta$ be $S^{n-1}$- and $S^{n-1}$-fibrations over $B$ respectively.

1.4.1. DEFINITION. We define $q(\xi, \eta)$ to be the least positive integer $q$ such that there exist nonnegative integers $r$ and $s$ with $m + r = n + s$ and a fibrewise map, $f: E_{\xi \oplus r} \rightarrow E_{\eta \oplus s}$, of degree $q$ at each fibre.

1.4.2. DEFINITION. $q(\xi) = q(\xi, 0)$.

1.4.3. Translation property. $q(\xi, \eta) = q(\xi - \eta, 0) = q(\xi - \eta)$.

**Proof.** Refer to §4.13 of [8].

The translation property makes clear that $q$ is not more general than $q$ and that it suffices to study $q$. We call $q$ the degree function on $K\overline{F}^{or}(B)$. We can alternatively define $q$ using Thom complexes. Let $\xi$ be an $S^{n-1}$-fibration over $B$.

1.4.4. DEFINITION. $q(\xi)$ is the least positive integer $q$ such that there exist a nonnegative integer $r$ and a map, $f: S^{r} B_{\xi} \rightarrow S^{n+r}$, of degree $q$ (i.e. the induced cohomology homomorphism satisfies $f^{*}(*) = q U_{r \oplus \xi}$ where $U_{r \oplus \xi} \in H^{n+r}(S^{r} B_{\xi})$ is the cohomology Thom class and $\iota \in H^{n+r}(S^{n+r})$ is the generator).

Let $c_{\xi}: E_{\xi \oplus 1} \rightarrow B_{\xi}$ be the quotient projection and $c_{\xi}^{*}: [B_{\xi}; S^{n}] \rightarrow [E_{\xi \oplus 1}; S^{n}]$ the induced map. The equivalence of Definitions 1.4.2 and 1.4.4 follows from the existence of a map $\iota_{\xi}: [E_{\xi}; S^{n-1}] \rightarrow [B_{\xi}; S^{n}]$ satisfying $c_{\xi}^{*} \circ \iota_{\xi} = \Sigma$ and $\iota_{\xi \oplus 1} \circ c_{\xi}^{*} = S$ where $\Sigma$ and $S$ are the appropriate suspension maps.
The degrees of maps \( f: S^r B^\xi \to S^n + r \) for all \( r \in \mathbb{Z}^+ \), form a subgroup of the integers, so \( q(\xi) \) divides the degree of any such map \( f \).

We note that \( q \) is related to the algebraic structure by the following:

1.4.5. \textbf{Nondegeneracy condition.} For \( u \in \widetilde{K}F^\text{or}(B) \), \( u = 0 \) iff \( q(u) = 1 \). The reader is referred to [8] for details and more information on \( q \).

1.5. \textbf{The homomorphism,} \( p: \widetilde{K}F^\text{or}(B) \to \widetilde{K}F^\text{or}(E) \), for a fibre bundle \( p: E \to B \). Let \( F \) be a closed smooth manifold and let \( p: E \to B \) define a fibre bundle with fibre \( F \).

1.5.1. \textbf{Proposition.} If \( u \in \widetilde{K}F^\text{or}(B) \) then \( q(p'(u))|q(u)|x(F)q(p(u)) \).

\textbf{Proof.} The first inequality (i.e. naturality) is obvious. The second follows from Lemma 1.3.1. For suppose \( u = \xi - n \) for an \( S^{n-1} \)-fibration \( \xi \) over \( B \) and \( q(p'(u)) = q_1 \). Let \( t_p(\xi): S^s B^\xi \to S^s E_p(\xi) \) be the transfer map defined in §1.3. If \( s \) is sufficiently large then there exists a map \( f: S^s E_p(\xi) \to S^{n+s} \), of degree \( q_1 \). Define \( g: S^s B^\xi \to S^{n+s} \) to be the composite \( g = f \circ t_p(\xi) \). Then if \( U_{s + \xi} \in H^{n+s}(S^s B^\xi) \), \( U_{s + \xi} D(p'(\xi)) \in H^{n+s}(S^s E_p(\xi)) \) are the cohomology Thom classes and \( \iota \in H^{n+s}(S^{n+s}) \) is the generator then by Lemma 1.3.1,

\[ g^*(\iota) = t_p^*(\xi) \circ f^*(\iota) = q_1 t_p^*(\xi)(U_{s + \xi} D(p'(\xi))) = \chi(F) q_1 U_{s + \xi} \chi(\iota). \]

Hence \( g \) is of degree \( \chi(F) q_1 \) and thus \( q(u)|\chi(F)q_1 \). Q.E.D.

1.5.2. \textbf{Corollary.} If \( \chi(F) = 1 \) then \( q(p'(u)) = q(u) \ \forall u \in \widetilde{K}F^\text{or}(B) \).

1.5.3 \textbf{Corollary.} If \( \chi(F) = 1 \) then \( p: \widetilde{K}F(B) \to \widetilde{K}F(E) \) is a monomorphism.

\textbf{Proof.} It follows from Corollary 1.5.2 and the nondegeneracy of \( q \).

1.6. \textbf{Proof of the Adams conjecture.} (i) \textit{for orientable bundles.} Let \( P \) be a principal \( SO(2n) \)-bundle over a finite dimensional CW-complex \( B \), \( E_\xi = P \times_{SO(2n)} R^{2n} \) the total space of the associated vector-bundle \( \xi \) and the projection and \( E_{p'(\xi)} = P \times_{N(T)} R^{2n} \). By Lemma 1.1.4, there exists a fibrewise map,

\[ f_k(p'(\xi)): E_{S(p'(\xi))} \to E_{S(\psi_k^k(p'(\xi)))}, \]

of degree \( k^n \) at each fibre and hence \( q((1 - \psi_k^k) p'(\xi)) = q(p'(\xi), \psi_k(p'(\xi))) \) divides \( k^n \). By Corollary 1.5.2 \( q((1 - \psi_k^k) p'(\xi)) = q((1 - \psi_k^k) p'(\xi)) \) divides \( k^n \). Hence by Dold’s Theorem mod \( k \) of \([2]\), the order of \( J((1 - \psi_k^k) \xi) \) in \( J(B) \) divides a power of \( k \).

(ii) \textit{for line bundles.} Let \( P \) be a principal \( \mathbb{Z}_2 \)-bundle over \( B \), \( E_\gamma = P \times_{\mathbb{Z}_2} R^1 \) and \( E_{c(\gamma)} = P \times_{\mathbb{Z}_2} S^1 \) be the total spaces of the associated line-bundle \( \gamma \) and the sphere-bundle \( S(c(\gamma)) \) of its complexification \( c(\gamma) \) respectively. Let \( f: S^1 \to S^1 \) be the degree 2 map given by \( f(z) = z^2, \forall z \in S^1 \). Then \( 1 \times f: P \times S^1 \to P \times S^1 \), passes to the quotient and defines a bundle map \( F: E_{S(c(\gamma))} = P \times_{\mathbb{Z}_2} S^1 \to (P / \mathbb{Z}_2) \times S^1 = B \times S^1 \) which is of degree 2 at each fibre. Hence by Dold’s Theorem mod \( k \), the order of \( J(rc(\gamma)) = J(2\gamma) = 2J(\gamma) \) is a power of 2 and thus the order of \( J(\gamma) \) is a power of 2.

\[ J((1 - \psi_k^k) \gamma) = \begin{cases} J(\gamma) & \text{if } k \text{ is even}, \\
0 & \text{if } k \text{ is odd}, \end{cases} \]
and if $k$ is even the order of $J((1 - \psi^k_R)\gamma)$ is a power of 2 and hence divides a power of $k$. If $k$ is odd the result trivially follows.

(iii) for nonorientable bundles. Let $\eta$ be a nonorientable vector-bundle over $B$ with first Stiefel Whitney class $w_1(\eta)$. Let $\gamma$ be the unique line bundle over $B$ such that $w_1(\gamma) = w_1(\eta)$. Let $\xi = \eta \oplus \gamma$, $w_1(\xi) = w_1(\eta) + w_1(\gamma) = 2w_1(\eta) = 0$ and hence $\xi$ is orientable. $J((1 - \psi^k_R)\eta) = J((1 - \psi^k_R)\xi) - J((1 - \psi^k_R)\gamma)$ and the order of the RHS divides a power of $k$ by (i) and (ii) above. Q.E.D.


2.1. Action of the Adams operations on $J(B)$. Let $J'(B)$ be the auxiliary quotient group of $K_R(B)$ defined in [4] as follows: For $u \in K_R(B)$, $J'(u) = 0$ iff $\rho_k^R(u) = a/\psi^k_R(a), \forall k \in \mathbb{Z}$ and for some $a \in K_R(B)$ of virtual dimension 1. It is a consequence of the Adams conjecture and Theorem 1.1 proved in Chapter 4 of [4] that the two groups $J(B)$ and $J'(B)$ are isomorphic.

2.1.1. Observation. The Adams operation $\psi^k_R$ passes to the quotient and acts on the group $J(B)$.

PROOF. Suppose $J(u) = 0$ for $u \in K_R(B)$. Then $J'(u) = 0$ and by definition, $\rho_k^R(u) = a/\psi^k_R(a), \forall k \in \mathbb{Z}$ and for some $a \in K_R(B)$ of virtual dimension 1. Applying $\psi^k_R$ to both sides we deduce that

$$\rho_k^R(\psi^k_R(u) = \frac{\psi^k_R(a)}{\psi^k_R(a)}, \forall k \in \mathbb{Z}$$

and for $\psi^k_R(a)$ of virtual dimension 1. Thus by definition, $J'(\psi^k_R(u)) = 0$ and thus $J(\psi^k_R(u)) = 0$. Q.E.D.

2.2. Preliminary lemmas.

2.2.1. Observation. Let $i = 0, 1, 2$ or $4$ (mod 8) and $x \in K_R(B)$ be of filtration $\geq i$. Then $\psi^k_R(x) = k^s x + x_{i_2}$ where $s = \max([i/2], 1), i_2 > i$ is the least positive integer which is congruent to 0, 1, 2 or 4 (mod 8) and $x_{i_2} \in K_R(B)$ is of filtration $\geq i_2$.

PROOF. It follows from the exact sequence of $K_R$-theory associated to the skeletal filtration of $B$ and the fact that $\psi^k_R = k^s 1$ on $K_R(S^i)$. Q.E.D.

2.2.2. LEMMA. Let $i = 0, 1, 2$ or $4$ (mod 8) and $x \in K_R(B)$ be of filtration $\geq i$. Then for any $n \in \mathbb{Z}^+$,

$$\psi^k_R(x) = k^{ns} x + k^{(n-1)s} x_{i_2} + \cdots + k^s x_{i_n} + x_{i_{n+1}}$$

where $s = \max([i/2], 1), i = i_1 < i_2 < \cdots < i_{n+1}$ is the ordered sequence of positive integers starting with $i$ and which are congruent to 0, 1, 2 or 4 (mod 8) and $x_{i_j} \in K_R(B)$ is of filtration $\geq i_j$ for $2 \leq j \leq n + 1$. Let $p$ be a prime and $J_p(B)$ denote the $p$-torsion of $J(B)$. If $J(x) \in J_p(B)$ then $J(x_{i_j}) \in J_p(B)$ for $1 \leq j \leq n + 1$.

PROOF. By induction on $n$. It is true for $n = 0$. Let $n \geq 0$ and assume it to be true for $n$. Apply $\psi^k_R$ to both sides of the equation. By Observation 2.2.1,

$$\psi^k_R(x) = k^s x + x_{i_2} \text{ and } \psi^k_R(x_{i_j}) = k^{[i_j/2]} x_{i_j} + x_{i_{j+1}} \text{ for } 2 \leq j \leq n + 1.$$ 

Let

$$y_{i_j} = x_{i_j} + k^{[i_j/2] - s} x_{i_j}, \quad 2 \leq j \leq n + 1,$$
and \( y_{i_{n+2}} = x_{i_{n+2}} \). Then \( y_{i_j} \in \tilde{K}_R(B) \) is of filtration \( \geq i_j \), \( 2 \leq j \leq n+2 \), and
\[
\psi_k^{k^{i_{j+1}}} (x) = k^{(n+1)^2} x + k^{n^2} y_{i_2} + k^{(n-1)^2} y_{i_3} + \cdots + y_{i_{n+2}}.
\]

If \( J(x) \in J_P(B) \) then \( J(x_{i_j}) \in J_P(B) \) by the induction hypothesis and \( J(\tilde{x}_{i_j}) \in J_P(B) \) by their definition and hence \( J(y_{i_j}) \in J_P(B) \), \( 2 \leq j \leq n+2 \).

2.2.3. Lemma. Let \( i = 0, 1, 2 \) or 4 (mod 8) and \( x \in \tilde{K}_R(B) \) be of filtration \( \geq i \). Then for any \( n \in \mathbb{Z}^+ \),
\[
k^{s+[i_2/2]+\cdots+[i_n/2]} x = \psi_k^k (x_{i_1}) + \psi_k^{k-1} (x_{i_2}) + \cdots + x_{i_{n+1}}
\]
where \( s = \max([i/2], 1) \) and \( i = i_1 < i_2 < \cdots < i_{n+1} \) is the ordered sequence of positive integers starting with \( i \) and which are congruent to 0, 1, 2 or 4 (mod 8) and \( x_{i_j} \in \tilde{K}_R(B) \) is of filtration \( \geq i_j \), \( 1 \leq j \leq n+1 \).

Proof. Obvious iteration of Observation 2.2.1 as in the proof of Lemma 2.2.2.

2.2.4. Corollary. Let \( i = 0, 1, 2 \) or 4 (mod 8) and \( x \in \tilde{K}_R(B) \) be of filtration \( \geq i \). Then for any \( n \in \mathbb{Z}^+ \),
\[
k^{s+[i_2/2]+\cdots+[i_n/2]} (1 - \psi_k^k x) = (\psi_k^k - \psi_k^{k-1}) x_{i_1} + \cdots + (1 - \psi_k^k) x_{i_{n+1}}
\]
where \( s = \max([i/2], 1) \), \( i = i_1 < i_2 < \cdots < i_{n+1} \) is the ordered sequence of positive integers starting with \( i \) and which are congruent to 0, 1, 2 or 4 (mod 8) and \( x_{i_j} \in \tilde{K}_R(B) \) is of filtration \( \geq i_j \), \( 1 \leq j \leq n+1 \).

Proof. Apply \( (1 - \psi_k^k) \) to both sides of the equation in Lemma 2.2.3.

2.3. Alternative formulations of the Adams conjecture. Using the lemmas of §2.2 we state three alternative formulations of the Adams conjecture and prove their equivalence.

2.3.1. Proposition. Let \( B \) be a finite dimensional CW-complex.
(i) \( \psi_k^k \) passes to the quotient and acts on the group \( J(B) \).
(ii) If \( u \in J(B) \) then the sequence \( (\psi_k^n(u))_{n \geq 1} \) stabilizes i.e. \( \psi_k^{n+1}(u) = \psi_k^n(u) \) for sufficiently large \( n \).

2.3.2. Proposition. (i) \( \psi_k^k \) passes to the quotient and acts on \( J(B) \).
(ii) Let \( p \) be a prime. The operator \( \psi_k^k \) is nilpotent on \( J_p(B) \) and is the identity on \( J_p(B) \) for \( p \neq p' \).

2.3.3. Proposition. (i) \( \psi_k^k \) passes to the quotient and acts on \( J(B) \).
(ii) Let \( p \) be a prime. The operator \( (1 - \psi_k^k) \) is an isomorphism on \( J_p(B) \) whose order is a power of \( p \) and is zero on \( J_{p'}(B) \) for \( p \neq p' \).

In what follows we adopt the notation that if \( n \) is an integer and \( p \) a prime then \( v_p(n) \) denotes the exponent of \( p \) in the prime factorization of \( n \).

2.3.4. Theorem. The Adams conjecture and Propositions 2.3.1, 2.3.2 and 2.3.3 are all equivalent.
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PROOF. (i) Adams conjecture ⇒ Proposition 2.3.2. The first part of 2.3.2 follows from Observation 2.1.1 which is a consequence of the Adams conjecture. Let \( u = J(x) \) for some \( x \in K_R(B) \) of filtration \( \geq i \). Since \( B \) is finite dimensional, there exists an integer \( d \) such that \( p^d v = 0 \) uniformly for \( v \in J_p(B) \). Choose \( n \geq 2 \max(d, \dim B) \). Since \( n \geq 2 \dim B \), the elements, \( x_{i \lfloor n/2 \rfloor + 1}, x_{i \lfloor n/2 \rfloor + 2}, \ldots, x_{i + 1} \) associated to \( \psi_R^n(x) \) by Lemma 2.2.2 are zero and hence

\[
\psi_R^n(x) = p^{n s} x + p^{(n-1)s} x_{i_2} + \cdots + p^{\lfloor (n+3)/2 \rfloor s} x_{i_{\lfloor n/2 \rfloor}}
\]

where \( s = \max(\lfloor i/2 \rfloor, 1), u = J(x) \in J_p(B) \) and hence by Lemma 2.2.2, \( J(x_{i_j}) \in J_p(B) \) for \( 1 \leq j \leq \lfloor n/2 \rfloor \). Since \( n \geq 2d \) the RHS is \( J \)-trivial. Thus \( J(\psi_R^n(x)) = 0 \) i.e. \( \psi_R^n(u) = 0 \). Finally, let \( v \in J_p(B) \) for \( p \neq p' \). Since \((1 - \psi_R^p)\) is linear, \((1 - \psi_R^p)v \in J_p(B)\). By the Adams conjecture \((1 - \psi_R^p)v \in J_p(B)\). Hence \((1 - \psi_R^p)v = 0\). Thus \((1 - \psi_R^p) = 0\) on \( J_p(B) \) or equivalently \( \psi_R^p = 1 \) on \( J_p(B) \).

(ii) Proposition 2.3.2 ⇒ Proposition 2.3.3. By Proposition 2.3.2, \( \psi_R^p = 0 \) on \( J_p(B) \) for some \( e \in Z^+ \). Assume also that \( p^d u = 0 \) uniformly for \( u \in J_p(B) \). Choose \( s = 1 + d + \sup_{1 \leq j \leq e - 1} v_p(j) \). Then on \( J_p(B) \)

\[
(1 - \psi_R^p)^{p^s} = \sum_{j=0}^{e-1} (-1)^j \binom{p^s}{j} \psi_R^p^j.
\]

It follows (essentially) from Lemma 6.1 of [5] that

\[
v_p \left( \binom{p^s}{j} \right) = s - v_p(j) > d
\]

and hence all the terms on the RHS except the first one are zero. Thus \((1 - \psi_R^p)^{p^s} = 1\) on \( J_p(B) \) and hence \((1 - \psi_R^p)\) is an isomorphism on \( J_p(B) \). That \((1 - \psi_R^p)\) is zero on \( J_p(B) \) for \( p \neq p' \) follows as in the proof of (i) above.

(iii) Proposition 2.3.3 ⇒ Proposition 2.3.1. By Proposition 2.3.3,

\[
(1 - \psi_R^p)^{p^s} = 1 + \sum_{j=1}^{p^s-1} (-1)^j \binom{p^s}{j} \psi_R^{p^j} + (-1)^{p^s} \psi_R^{p^s} = 1.
\]

Since

\[
p \binom{p^s}{j} \quad \text{for } 1 \leq j \leq p^s - 1
\]

we can write

\[
\sum_{j=1}^{p^s-1} (-1)^j \binom{p^s}{j} \psi_R^{p^j} = (-1)^{p^s-1} p T
\]

for some operator \( T \) on \( J_p(B) \). Thus \( \psi_R^{p^s} = pT \) on \( J_p(B) \). Since \( p^d u = 0 \) uniformly for \( u \in J_p(B) \), raising both sides to their dth-power yields, \( \psi_R^{p^s} = p^d T^d = 0 \) i.e. \( \psi_R^p \) is nilpotent on \( J_p(B) \). Let \( u \in J(B) \). By Proposition 2.3.2, \((1 - \psi_R^p)u \in J_p(B)\) and thus \((\psi_R^{p^s} - \psi_R^{p^s+1})u = \psi_R^{p^s}(1 - \psi_R^p)u = 0\) for sufficiently large \( n \). Finally, let \( k \in Z \) and \( k = p_1 p_2 \cdots p_t \) for primes \( p_t \).

\[
(\psi_R^k - \psi_R^{k+1})u = (\psi_R^{p_1} - \psi_R^{p_1+1})\psi_R^{p_2} \cdots p_t(u) + (\psi_R^{p_2} - \psi_R^{p_2+1})\psi_R^{p_3} \cdots p_t(u) + \cdots + (\psi_R^{p_t} - \psi_R^{p_t+1})\psi_R^{p_t+1} p_t+1 \cdots p_t+1(u) = 0.
\]
Thus $\psi^n_R(u) = \psi^{n+1}_R(u)$ for sufficiently large $n$.

(iv) Proposition 2.3.1 ⇒ Adams conjecture. By Proposition 2.3.1, there exists a positive integer $N$ such that $(\psi^n_R - \psi^{n+1}_R)u = 0$ for $n \geq N$ and $\forall u \in J(B)$. Let $x \in \tilde{K}_R(B)$ be of filtration $\geq i$, $s = \max([i/2], 1)$ and choose $n \geq 2 \max(N, \dim B)$. Since $n \geq 2 \dim B$, the elements $x_{i[n/2]+1}, \ldots, x_{i[n/2]+1}$ associated to $x$ by Corollary 2.2.4 are zero. Let $e = s + [i_2/2] + \cdots + [i_n/2]$. By Corollary 2.2.4,

$$k^e(1 - \psi^n_R)x = (\psi^n_R - \psi^{n+1}_R)x_1 + (\psi^{n-1}_R - \psi^n_R)x_{i_2} + \cdots + (\psi^{k[(n+3)/2]}_R - \psi^{k[(n+5)/2]}_R)x_{i[n/2]}.$$ 

Since $n/2 \geq N$, the RHS is $J$-trivial. Thus $k^e J((1 - \psi^n_R)x) = 0$ for sufficiently large $e$. Q.E.D.

REFERENCES


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