

## THE DETERMINATION OF ALL VARIETIES CONSISTING OF ABSOLUTELY CLOSED SEMIGROUPS

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**ABSTRACT.** An example is given of a right normal band that is not absolutely closed. This enables us to determine all varieties of semigroups that consist entirely of absolutely closed semigroups.

**1. Introduction** Let  $U, S$  be semigroups with  $U \subseteq S$ . Following Howie and Isbell [4] we say that  $U$  dominates an element  $d \in S$  if for every semigroup  $T$  and all homomorphisms  $\alpha, \beta: S \rightarrow T$ ,  $\alpha|U = \beta|U$  implies  $d\alpha = d\beta$ . The set of all elements of  $S$  dominated by  $U$  is called the *dominion* of  $U$  in  $S$  and is denoted by  $\text{Dom}(U, S)$ . It is easy to see that  $\text{Dom}(U, S)$  is a subsemigroup of  $S$  containing  $U$ . We say that  $U$  is *closed* in  $S$  if  $\text{Dom}(U, S) = U$  and  $U$  is *absolutely closed* if  $U$  is closed in every containing semigroup  $S$ . We call a variety of semigroups  $\mathcal{V}$  *absolutely closed* if  $U \in \mathcal{V}$  implies  $U$  is absolutely closed.

Semigroup dominions are characterised by the following celebrated theorem.

**RESULT 1 (ISBELL'S ZIGZAG THEOREM [5, THEOREM 2.3]).** *Let  $U$  be a subsemigroup of a semigroup  $S$  and let  $d \in S$ . Then  $d \in \text{Dom}(U, S)$  if and only if  $d \in U$  or there is a series of factorizations of  $d$  as follows:*

$$d = u_0 y_1 = x_1 u_1 y_1 = x_1 u_2 y_2 = x_2 u_3 y_2 = \cdots = x_m u_{2m-1} y_m = x_m u_{2m}$$

where  $m \geq 1$ ,  $u_i \in U$ ,  $x_i, y_i \in S$  with  $u_0 = x_1 u_1$ ,  $u_{2i-1} y_i = u_{2i} y_{i+1}$ ,  $x_i u_{2i} = x_{i+1} u_{2i+1}$  ( $1 \leq i \leq m-1$ ) and  $u_{2m-1} y_m = u_{2m}$ .

Such a series of factorizations is called a *zigzag* in  $S$  over  $U$  with *value*  $d$  and *length*  $m$ . In this paper we require just the "if" part of the statement of the zigzag theorem, a proof of which is an easy manipulation of the zigzag.

We use, whenever possible, and usually without comment, the notations and conventions of Clifford and Preston [1] or Howie [3].

**2. The example.** We give an example of a four-element right normal band which is not closed in a containing semigroup of seven elements.

Let  $T_6$  denote the full transformation semigroup on  $\{1, 2, 3, 4, 5, 6\}$ . We denote a particular member of  $T_6$  by a list of the images of the domain elements  $1, 2, \dots, 6$ . Let  $S$  be the subsemigroup of  $T_6$  generated by the maps:  $a_1 = (1\ 1\ 1\ 4\ 4\ 4)$ ,  $a_2 = (1\ 1\ 3\ 4\ 4\ 6)$ ,  $a_3 = (2\ 2\ 3\ 5\ 5\ 6)$ ,  $x = (3\ 3\ 3\ 6\ 6\ 6)$ ,  $y = (3\ 2\ 3\ 6\ 5\ 6)$ . Let  $U$  be the subsemigroup of  $S$  generated by  $\{a_1, a_2, a_3\}$ . The multiplication table for  $S$  is given below.

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	$a_1$	$a_1a_3$	$a_2$	$a_3$	$x$	$y$	$ya_2$
$a_1$	$a_1$	$a_1a_3$	$a_1$	$a_1a_3$	$x$	$x$	$x$
$a_1a_3$	$a_1$	$a_1a_3$	$a_1$	$a_1a_3$	$x$	$a_1a_3$	$a_1$
$a_2$	$a_1$	$a_1a_3$	$a_2$	$a_3$	$x$	$x$	$x$
$a_3$	$a_1$	$a_1a_3$	$a_2$	$a_3$	$x$	$a_3$	$a_2$
$x$	$a_1$	$a_1a_3$	$x$	$x$	$x$	$x$	$x$
$y$	$a_1$	$a_1a_3$	$ya_2$	$y$	$x$	$y$	$ya_2$
$ya_2$	$a_1$	$a_1a_3$	$ya_2$	$y$	$x$	$x$	$x$

The semigroup  $U = \{a_1, a_2, a_3, a_1a_3\}$  and is a right normal subband of  $S$ . Furthermore  $x \in \text{Dom}(U, S) \setminus U$  since the following is a zigzag in  $S$  over  $U$  with value  $x$ :

$$x = a_1y = xa_1y = xa_2y = xa_3y = xa_3$$

and

$$a_1 = xa_1, a_1y = a_2y, xa_2 = xa_3, a_3y = a_3.$$

This example and its left-right dual show that the varieties of right normal bands and of left normal bands are not absolutely closed.

**3. The absolutely closed varieties.** All absolutely closed varieties of semigroups contain none of the following semigroups:

- A, a three-element null semigroup,
- B, a  $2 \times 2$  rectangular band,
- C, a two-element right zero semigroup with adjoined zero,
- D, the right-left dual of C.

This is because  $A$  is not absolutely closed [2] and nor is  $B$  [4], while inspection of the lattice of bands [3, p. 124] shows that  $C$  (respectively  $D$ ) generates the variety of right (respectively left) normal bands.

**THEOREM 2.** *The absolutely closed varieties of semigroups are exactly the varieties consisting entirely of semilattices of groups, or entirely of right groups or entirely of left groups.*

**REMARK.** These varieties are the subvarieties of those defined by identities of the form:  $[x = x^{1+n}, x^n y^n = y^n x^n, \text{ some } n \geq 1]$  (semilattices of groups),  $[x = x^{1+n}, x^n y = y \text{ for some } n \geq 1]$  (right groups) and  $[x = x^{1+n}, y x^n = y \text{ for some } n \geq 1]$  (left groups).

**PROOF OF THEOREM.** The varieties listed in the statement of the theorem consist entirely of inverse semigroups, of right simple semigroups and of left simple semigroups respectively. These three types of semigroups are absolutely closed [4]. Hence the listed varieties are absolutely closed.

Conversely suppose  $\mathcal{V}$  is an absolutely closed variety of semigroups. The three-element null semigroup  $A$  satisfies all nontrivial identities except those for which one side consists of a single variable  $x$ . Any such identity implies an identity of the kind  $x = x^{1+n}$  for some  $n \geq 1$ . Since  $A$  is excluded from  $\mathcal{V}$ ,  $\mathcal{V}$  therefore admits

an identity of the form  $x = x^{1+n}$  for some  $n \geq 1$ . Hence each  $S \in \mathcal{V}$  is a union of groups and so is a semilattice of completely simple semigroups [3, Chapter 4, Theorem 1.7].

Let  $D$  be any one of the completely simple semigroups occurring as a  $\mathcal{D}$ -class in  $S$ . If  $D$  is neither right simple nor left simple then  $D/\mathcal{H}$  (where  $\mathcal{H}$  is Green's  $\mathcal{H}$  relation on  $D$ ) is a rectangular band which contains a  $2 \times 2$  rectangular subband; this implies  $B \in \mathcal{V}$ , a contradiction. Therefore  $D$  is a right group or a left group. However, if  $\mathcal{V}$  contained a right group and a left group, both of which were not groups, then  $\mathcal{V}$  would contain  $R$  and  $L$ , a two-element right and a two-element left zero semigroup. This would imply that  $B \simeq R \times L \in \mathcal{V}$ , a contradiction. Hence  $\mathcal{V}$  consists entirely of semilattices of right groups or entirely of semilattices of left groups.

Suppose  $\mathcal{V}$  consists entirely of semilattices of right groups. Then either  $\mathcal{V}$  consists entirely of semilattices of groups, entirely of right groups or  $\mathcal{V}$  contains a nontrivial semilattice and a right group which is not a group. In the latter case  $\mathcal{V}$  contains a two-element semilattice  $T = \{0, e\}$  and a two-element right zero semigroup  $R = \{f, g\}$ . The product  $R \times T$  has an ideal  $I = \{(f, 0), (g, 0)\}$ . The factor semigroup  $(R \times T)/I \simeq C$ , implying  $C \in \mathcal{V}$ , a contradiction. Hence this alternative does not arise and so  $\mathcal{V}$  consists entirely of semilattices of groups or entirely of right groups. The dual argument shows that if  $\mathcal{V}$  consists entirely of semilattices of left groups then in fact  $\mathcal{V}$  consists entirely of semilattices of groups or entirely of left groups. This completes the proof of the theorem.

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