AN EXTRAPOLATION THEOREM IN THE THEORY OF $A_p$ WEIGHTS

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ABSTRACT. A new proof is given of the extrapolation theorem of J. L. Rubio de Francia [7]. Unlike the proof in [7], the proof presented here is independent of the theory of vector valued inequalities.

I am indebted to J. L. Rubio de Francia for calling my attention to the following:

**THEOREM.** Let $T$ be a sublinear operator defined on a class of measurable functions in $\mathbb{R}^n$. Suppose that, for some $p_0$ with $1 \leq p_0 < \infty$, and for every weight $w$ in the class $A_{p_0}$ of B. Muckenhoupt (see [1 and 5]), $T$ satisfies an inequality

$$\int_{\mathbb{R}^n} |Tf(x)|^{p_0} w(x) \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^{p_0} w(x) \, dx$$

for every $f$, where $C = C_{p_0}(w)$ depends only on the $A_{p_0}$ constant for $w$. Then, for every $p$ with $1 < p < \infty$, and every weight $w$ in the class $A_p$, $T$ satisfies an inequality

$$\int_{\mathbb{R}^n} |Tf(x)|^{p} w(x) \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^{p} w(x) \, dx$$

for every $f$, where $C = C_p(w)$ depends only on the $A_p$ constant for $w$.

The original proof is given in [7]. It is based upon the equivalence between weighted inequalities and vector valued inequalities obtained in [6]. The purpose of the present note is to give a new proof of the theorem stated above. This new proof is independent of vector valued inequalities and lies completely within the realm of the theory of $A_p$ weights.

The proof will follow after two lemmas. The letter $C$ will be used to denote a constant, not necessarily the same at each occurrence, as has been done already in the statement of the theorem.

**LEMMA 1.** Let $1 < p < \infty$, $w \in A_p$, $0 < t < 1$. For a function $g > 0$, define $G = \{M(g^{1/t}w^{-1}w^t)^t\}$, where $M$ stands for the Hardy-Littlewood maximal operator. Then

(i) $g \mapsto G$ is an operator bounded in $L^{p'/t}(w)$, where $p' = p/(p-1)$, the exponent conjugate to $p$;

(ii) $(gw, Gw) \in A_{p-tp/p'}$, the class of pairs of weights appearing in [5].

Besides, the operator norm in (i) and the constant for the pair of weights in (ii) depend only on the $A_p$ constant for $w$.

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Proof. (i) 
\[
\|G\|_{p'/t,w}^{p'/t} = \int_{\mathbb{R}^n} |G(x)|^{p'/t} w(x) \, dz = \int_{\mathbb{R}^n} |M(g^{1/t} w)|^{p'} w^{1-t'p'} 
\leq C \int_{\mathbb{R}^n} g^{p'/t} w^{p'} w^{1-t'} = C \|g\|_{p'/t,w}^{p'/t},
\]
since \( w^{1-t'} = w^{-1/(p-1)} \in A_{p'}.

(ii) Let \( p_0 = p - tp/p' \). Clearly \( p_0 \geq 1 \), since \( p_0 - 1 = (1-t)p/p' \geq 0 \). If \( t = 1 \), then \( p_0 = 1 \) and it is immediate that \( (gw, Gw) \in A_1 \) with constant 1, because \( M(gw) = Gw \). Let \( 0 < t < 1 \). We have to prove that
\[
\left( \frac{1}{|Q|} \int_Q g^w \right) \left( \frac{1}{|Q|} \int_Q (Gw)^{-1/(p-1)} \right)^{p_0 - 1} \leq C
\]
with a constant \( C \) independent of the cube \( Q \), where \( |Q| \) stands for the Lebesgue measure of \( Q \). The left-hand side of this inequality is
\[
\left( \frac{1}{|Q|} \int_Q g^w \right) \left( \frac{1}{|Q|} \int_Q M(g^{1/t} w)^{-tp'/p'(1-t)} w^{-p'/p} \right)^{(1-t)p/p'}.
\]

By using Hölder’s inequality with exponents \( 1/t \) and its conjugate \( 1/(1-t) \), we see that the last line is bounded by
\[
\left( \frac{1}{|Q|} \int_Q g^{1/t} w \right)^t \left( \frac{1}{|Q|} \int_Q w \right)^{1-t} \left( \frac{1}{|Q|} \int_Q (g^{1/t} w)^{-tp'/p'(1-t)} w^{-p'/p} \right)^{(1-t)p/p'}
\leq \left( \frac{1}{|Q|} \int_Q g^{1/t} w \right)^t \left( \frac{1}{|Q|} \int_Q w \right)^{1-t} \left( \frac{1}{|Q|} \int_Q g^{1/t} w \right)^{-t}
\times \left( \frac{1}{|Q|} \int_Q w^{-(1-t)} \right)^{(p-1)(1-t)} \leq C
\]
where \( C \) is precisely the \( A_p \) constant for \( w \), raised to the power \( 1-t \). This finishes the proof.  

OBSERVATION. If we set \( p - tp/p' = p_0 < p \), we shall have \( 1 - t/p' = p_0/p \), or, in other words, \( p'/t = (p/p_0) \). With this change of notation, Lemma 1 can be restated as follows: Let \( 1 \leq p_0 < p \), \( w \in A_{p_0} \). Then, for every \( g \geq 0 \) belonging to \( L^{(p/p_0)}(w) \), there is a \( G \geq g \) such that \( \|G\|_{(p/p_0),w} \leq C\|g\|_{(p/p_0),w} \) and \( Gw \in A_{p_0} \), with both \( C \) and the \( A_{p_0} \) constant for the pair depending only on the \( A_p \) constant for \( w \).

Now Lemma 1 can be used to obtain a stronger result:

Lemma 2. (a) If \( 1 \leq p_0 < p \) and \( w \in A_{p_0} \), then, for every \( g \geq 0 \) belonging to \( L^{(p/p_0)}(w) \), there is a \( G \geq g \) such that \( \|G\|_{(p/p_0),w} \leq C\|g\|_{(p/p_0),w} \) and \( Gw \in A_{p_0} \), with both \( C \) and the \( A_{p_0} \) constant for \( Gw \) depending only on the \( A_p \) constant for \( w \).

(b) If \( 1 < p < p_0 \) and \( w \in A_{p_0} \), then, for every \( g \geq 0 \) belonging to \( L^{p/(p_0-p)}(w) \), there is a \( G \geq g \) such that \( \|G\|_{p/(p_0-p),w} \leq C\|g\|_{p/(p_0-p),w} \) and \( G^{-1} w \in A_{p_0} \), with both \( C \) and the \( A_{p_0} \) constant for \( G^{-1} w \) depending only on the \( A_p \) constant for \( w \).

Proof. (a) Let \( g_0 = g \). According to the observation made, there is \( g_1 \geq g \) such that \( \|g_1\|_{(p/p_0),w} \leq C\|g\|_{(p/p_0),w} \) and the inequality,
\[
\int_{\{x : Mf(x) > s\}} g_0 w \leq C s^{-p_0} \int_{\mathbb{R}^n} |f(x)|^{p_0} g_1 w,
\]
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holds for every function \( f \) and every \( s > 0 \), with \( C \) depending only on the \( A_p \) constant for \( w \). Then we proceed by induction. We can use \( g_1 \) in place of \( g_0 \) and continue in this way. In general, given \( g_j \), we obtain \( g_{j+1} \geq g_j \) such that \( \|g_{j+1}\|_{(p/p_0)'},w \leq C\|g_j\|_{(p/p_0)'},w \leq \cdots \leq C^{j+1}\|g\|_{(p/p_0)'},w \) and the inequality

\[
\int_{\{x : Mf(x) > s\}} g_{j+1}w \leq Cs^{-p_0} \int_{\mathbb{R}^n} |f(x)|^{p_0} g_{j+1}w,
\]

holds for every function \( f \) and every \( s > 0 \), with \( C \) depending only on the \( A_p \) constant for \( w \). Now consider \( G = \sum_{j=0}^{\infty} (C+1)^{-j}g_j \) with \( C \) the same as above. Since \( (C+1)^{-j}\|g_j\|_{(p/p_0)'},w \leq (C/(C+1))^j\|g\|_{(p/p_0)'},w \), the series converges in \( L^{(p/p_0)'}(w) \) and we get \( G \geq g \) with \( \|G\|_{(p/p_0)'},w \leq (C+1)\|g\|_{(p/p_0)'},w \) and also, adding in \( j \) the inequalities (3),

\[
\int_{\{x : Mf(x) > s\}} Gw \leq (C+1)s^{-p_0} \int_{\mathbb{R}^n} |f(x)|^{p_0} Gw,
\]

for every \( s > 0 \); in other words: \( Gw \in A_{p_0} \) and its \( A_{p_0} \) constant depends only on the \( A_p \) constant for \( w \).

(b) Now \( 1 < p < p_0 \) and \( w \in A_p \). This is the same as saying that \( 1 < p_0 < p' \) and \( u \equiv w^{-1/(p-1)} \in A_p' \). Therefore, we can apply part (a) to conclude that, for every \( h \geq 0 \) belonging to \( L^{(p'/p_0)'}(u) \), there is an \( H \geq h \) such that \( \|H\|_{(p'/p_0)',u} \leq C\|h\|_{(p'/p_0)'},u \) and \( H \in A_{p_0} \), with \( C \) and the \( A_{p_0} \) constant for \( H \) depending only on the \( A_p \) constant for \( w \). But \( (p'/p_0)' = (p_0 - 1)p/(p_0 - p) \), so that \( h \in L^{(p'/p_0)'}(u) \) if and only if \( h^{p_0-1}w^{-(p_0-p)/(p-1)} \in L^p/(p_0-p)(w) \). Also \( H \in A_{p_0} \) if and only if \( ( Hu )^{-1/(p_0-1)} = [ H^{p_0-1} w^{-(p_0-p)/(p-1)} ]^{-1} w \in A_{p_0} \). Thus, if we are given \( g \geq 0 \) in \( L^{p/(p_0-p)}(w) \), we just write \( g = h^{p_0-1}w^{-(p_0-p)/(p-1)} \) with \( h \in L^{(p'/p_0)'}(u) \), obtain the corresponding \( H \) and then define \( G = H^{p_0-1}w^{-(p_0-p)/(p-1)} \). This proves part (b). \( \square \)

We are ready to give the

**Proof of the Theorem.**

(a) Let \( 1 \leq p_0 < p, w \in A_p \) and \( f \in L^p(w) \),

\[
\|Tf\|_{p_0,w} = \|Tf^{p_0}\|_{p_0,p_0,w} = \int_{\mathbb{R}^n} |Tf|^{p_0} g_{p_0,w}
\]

for some \( g \geq 0 \) with \( \|g\|_{(p_0)'/w} = 1 \). Associate with \( g \) a function \( G \) as in Lemma 2. Then, the last integral is bounded by

\[
\int_{\mathbb{R}^n} |Tf|^{p_0} Gw \leq C \int_{\mathbb{R}^n} |f|^{p_0} Gw \leq C\|f\|_{p_0,p_0,w} \|G\|_{(p_0)'},w \leq C\|f\|_{p,p,w}.
\]

(b) Let \( 1 < p < p_0, w \in A_p \) and \( f \in L^p(w) \),

\[
\|f\|_{p_0,w} = \|f^{p_0}\|_{p_0,p_0,w} = \int_{\mathbb{R}^n} |f|^{p_0} g_{-1}w
\]

for some \( g \geq 0 \) such that \( \|g\|_{p/(p_0-p),w} = 1 \). Associate with \( g \) a function \( G \) as in part (b) of Lemma 2. Then

\[
\|Tf\|_{p_0,w} = \|Tf^{p_0}\|_{p_0,p_0,w} \leq \left( \int_{\mathbb{R}^n} |Tf|^{p_0} G_{-1}w \right) \|G\|_{p/(p_0-p),w} \leq C \int_{\mathbb{R}^n} |f|^{p_0} G_{-1}w \leq C \int_{\mathbb{R}^n} |f|^{p_0} g_{-1}w = C\|f\|_{p,p,w}.
\]

This ends the proof. \( \square \)
OBSERVATION. There is a weak-type version of the theorem, in which one substitutes for (1) the inequality
\[ \int_{\{x : |Tf(x)| > s\}} w(x) dx \leq Cs^{-p_0} \int_{R^n} |f(x)|^{p_0} w(x) dx, \quad s > 0, \]
and also changes the inequality (2) in the conclusion to
\[ \int_{\{x : |Tf(x)| > s\}} w(x) dx \leq Cs^{-p} \int_{R^n} |f(x)|^p w(x) dx, \quad s > 0. \]

The proof of this weak-type version is as follows. For \( s > 0 \), let us call \( E_s = \{x : |Tf(x)| > s\} \). For a set \( E \), \( w(E) \) will stand for \( \int_E w(x) dx \) and \( \chi_E \) will denote the characteristic function of \( E \).

(a) Let \( 1 < p_0 < p, w \in A_p \) and \( f \in L^p(w) \). Then, for \( s > 0 \),
\[ sp_0 w(E_s)^{p_0/p} = sp_0 \|\chi_{E_s}\|_{p/p_0,w} = sp_0 \int_{R^n} \chi_{E_s} g w \]
for some \( g \geq 0 \) with \( \|g\|_{(p_0/p,w)} = 1 \). Associate with \( g \) a function \( G \) as in Lemma 2, part (a). Then
\[ sp_0 w(E_s)^{p_0/p} \leq sp_0 \int_{R^n} \chi_{E_s} G w \leq C \int_{R^n} |f|^{p_0} G w \leq C \|f\|_{p_0,w}^{p_0}. \]
Thus, we have proved that \( w(E_s) \leq Cs^{-p} \|f\|_{p,w}^p \), which is precisely (2').

(b) Let \( 1 < p < p_0, w \in A_p \) and \( f \in L^p(w) \). Choose \( g \) as in the proof of the corresponding part of the strong type theorem. Associate with \( g \) the function \( G \) as in Lemma 2 part (b). Then, for \( s > 0 \),
\[ sp_0 w(E_s)^{p_0/p} = sp_0 \|\chi_{E_s}\|_{p/p_0,w} \leq C \int_{R^n} \chi_{E_s} G^{-1} w \]
\[ \leq C \int_{R^n} |f|^{p_0} G^{-1} w \leq C \int_{R^n} |f|^{p_0} g^{-1} w = C \|f\|_{p,w}^{p_0}, \]
which gives (2') as before.

The two theorems we have proved are also valid for the weights associated to other maximal operators. Some interesting examples are given in [7]. Our approach works for all of them because the description of the classes \( A_p \) is formally the same. These general weights do not necessarily satisfy a reverse Hölder’s inequality. However, in the particular situation we are dealing with, the \( A_p \) weights do satisfy a reverse Hölder’s inequality (see [1 and 5]). This fact allows us to strengthen the weak type theorem so that the strong type theorem becomes a corollary. Indeed, we can get the strong type inequalities (2) from the weak type inequalities (1'). We just need to realize that, because of the reverse Hölder’s inequality, if \( w \in A_p \) and \( 1 < p < \infty \), there is \( \varepsilon > 0 \) such that \( w \in A_{p-\varepsilon} \) also. Then, Marcinkiewicz interpolation theorem applies, and we obtain the strong type estimate.

The case \( p = 1 \) has been excluded. In fact, it has to be excluded. We can always extrapolate from \( p_0 = 1 \) to any \( p \); but extrapolation down to \( p = 1 \) is not legitimate. For example, the maximal partial sum operator of Fourier series is bounded in \( L^p(w) \) whenever \( 1 < p < \infty \) and \( w \in A_p \) (see [3]), but it fails to be of weak type 1, 1 with respect to Lebesgue measure.

We should mention that the case \( t = 1 \) of our Lemma 1 has been used by R. Coifman, P. Jones and J. L. Rubio de Francia [2] to give a simple constructive
proof of the factorization theorem of P. Jones [4]. Their proof simplifies a previous (nonconstructive) proof by J. L. Rubio de Francia [7], which used vector valued inequalities.

REFERENCES


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