AN EXTRAPOLATION THEOREM
IN THE THEORY OF $A_p$ WEIGHTS

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ABSTRACT. A new proof is given of the extrapolation theorem of J. L. Rubio de Francia [7]. Unlike the proof in [7], the proof presented here is independent of the theory of vector valued inequalities.

I am indebted to J. L. Rubio de Francia for calling my attention to the following:

**THEOREM.** Let $T$ be a sublinear operator defined on a class of measurable functions in $\mathbb{R}^n$. Suppose that, for some $p_0$ with $1 \leq p_0 < \infty$, and for every weight $w$ in the class $A_{p_0}$ of B. Muckenhoupt (see [1 and 5]), $T$ satisfies an inequality

$$
\int_{\mathbb{R}^n} |Tf(x)|^{p_0}w(x)\,dx \leq C \int_{\mathbb{R}^n} |f(x)|^{p_0}w(x)\,dx
$$

for every $f$, where $C = C_{p_0}(w)$ depends only on the $A_{p_0}$ constant for $w$. Then, for every $p$ with $1 < p < \infty$, and every weight $w$ in the class $A_p$, $T$ satisfies an inequality

$$
\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\,dx \leq C \int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx
$$

for every $f$, where $C = C_p(w)$ depends only on the $A_p$ constant for $w$.

The original proof is given in [7]. It is based upon the equivalence between weighted inequalities and vector valued inequalities obtained in [6]. The purpose of the present note is to give a new proof of the theorem stated above. This new proof is independent of vector valued inequalities and lies completely within the realm of the theory of $A_p$ weights.

The proof will follow after two lemmas. The letter $C$ will be used to denote a constant, not necessarily the same at each occurrence, as has been done already in the statement of the theorem.

**LEMMA 1.** Let $1 < p < \infty$, $w \in A_p$, $0 < t \leq 1$. For a function $g > 0$, define $G = \{M(g^{1/t}w)^{-1} \}^t$, where $M$ stands for the Hardy-Littlewood maximal operator. Then

(i) $g \mapsto G$ is an operator bounded in $L^{p'/t}(w)$, where $p' = p/(p-1)$, the exponent conjugate to $p$;

(ii) $(gw, Gw) \in A_{p-tp'/p'}$, the class of pairs of weights appearing in [5].

Besides, the operator norm in (i) and the constant for the pair of weights in (ii) depend only on the $A_p$ constant for $w$. 

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PROOF. (i)

\[ ||G||_{p'/t,w}^{p'/t} = \int_{\mathbb{R}^n} |G(x)|^{p'/t} w(x) dz = \int_{\mathbb{R}^n} |M(g^{1/t}w)|^{p'/t} w^{1-p'} \]

\[ \leq C \int_{\mathbb{R}^n} g^{p'/t} w^{p'} w^{1-p'} = C ||g||_{p'/t,w}^{p'/t} \]

since \( w^{1-p'} = w^{-1/(p-1)} \in A_p' \).

(ii) Let \( p_0 = p - tp'/p' \). Clearly \( p_0 \geq 1 \), since \( p_0 - 1 = (1-t)p/p' \geq 0 \). If \( t = 1 \), then \( p_0 = 1 \) and it is immediate that \((gw,Gw) \in A_1\) with constant 1, because \( M(gw) = Gw \). Let \( 0 < t < 1 \). We have to prove that

\[ \left( \frac{1}{|Q|} \int_Q gw \right) \left( \frac{1}{|Q|} \int_Q (Gw)^{-1/(p_0-1)} \right)^{p_0-1} \leq C \]

with a constant \( C \) independent of the cube \( Q \), where \( |Q| \) stands for the Lebesgue measure of \( Q \). The left-hand side of this inequality is

\[ \left( \frac{1}{|Q|} \int_Q g w \right) \left( \frac{1}{|Q|} \int_Q M(g^{1/t}w)^{-tp'/(p(1-t))} w^{-p'/p'} \right)^{1-t} \]

By using Hölder’s inequality with exponents \( 1/t \) and its conjugate \( 1/(1-t) \), we see that the last line is bounded by

\[ \left( \frac{1}{|Q|} \int_Q g^{1/t} w \right)^t \left( \frac{1}{|Q|} \int_Q w \right)^{1-t} \left( \frac{1}{|Q|} \int_Q M(g^{1/t}w)^{-tp'/(p(1-t))} w^{-1/(p-1)} \right)^{(1-t)p/p'} \]

\[ \leq \left( \frac{1}{|Q|} \int_Q g^{1/t} w \right)^t \left( \frac{1}{|Q|} \int_Q w \right)^{1-t} \left( \frac{1}{|Q|} \int_Q g^{1/t} w \right)^{-(p-1)(1-t)} \]

\[ \leq C \]

where \( C \) is precisely the \( A_p \) constant for \( w \), raised to the power \( 1 - t \). This finishes the proof. \( \square \)

OBSERVATION. If we set \( p - tp'/p' = p_0 < p \), we shall have \( 1 - t/p' = p_0/p \), or, in other words, \( p'/t = (p/p_0)^{-1} \). With this change of notation, Lemma 1 can be restated as follows: Let \( 1 \leq p_0 < p \), \( w \in A_{p_0} \). Then, for every \( g \geq 0 \) belonging to \( L^{(p/p_0)}(w) \), there is a \( G \geq g \) such that \( ||G||_{(p/p_0),w} \leq C ||g||_{(p/p_0),w} \) and \( Gw \in A_{p_0} \), with both \( C \) and the \( A_{p_0} \) constant for the pair depending only on the \( A_p \) constant for \( w \).

Now Lemma 1 can be used to obtain a stronger result:

**Lemma 2.** (a) If \( 1 \leq p_0 < p \) and \( w \in A_{p_0} \), then, for every \( g \geq 0 \) belonging to \( L^{(p/p_0)}(w) \), there is a \( G \geq g \) such that \( ||G||_{(p/p_0),w} \leq C ||g||_{(p/p_0),w} \) and \( Gw \in A_{p_0} \), with both \( C \) and the \( A_{p_0} \) constant for \( Gw \) depending only on the \( A_p \) constant for \( w \). (b) If \( 1 < p < p_0 \) and \( w \in A_{p_0} \), then, for every \( g \geq 0 \) belonging to \( L^p/(p_0-p)(w) \), there is a \( G \geq g \) such that \( ||G||_{p/(p_0-p),w} \leq C ||g||_{p/(p_0-p),w} \) and \( G^{-1} w \in A_{p_0} \), with both \( C \) and the \( A_{p_0} \) constant for \( G^{-1} w \) depending only on the \( A_p \) constant for \( w \).

**Proof.** (a) Let \( g_0 = g \). According to the observation made, there is \( g_1 \geq g \) such that \( ||g_1||_{(p_0-p),w} \leq C ||g||_{(p_0-p),w} \) and the inequality,

\[ \int_{\{x : Mf(x) > s\}} g_0 w \leq C s^{-p_0} \int_{\mathbb{R}^n} |f(x)|^{p_0} g_1 w, \]
holds for every function $f$ and every $s > 0$, with $C$ depending only on the $A_p$ constant for $w$. Then we proceed by induction. We can use $g_1$ in place of $g_0$ and continue in this way. In general, given $g_j$, we obtain $g_{j+1} \geq g_j$ such that $\|g_{j+1}\|_{(p/p_0)',w} \leq C\|g_j\|_{(p/p_0)',w} \leq \cdots \leq C^{j+1}\|g\|_{(p/p_0)',w}$ and the inequality

$$\int_{\{x : Mf(x) > s\}} g_{j+1}w \leq Cs^{-p_0} \int_{\mathbb{R}^n} |f(x)|^{p_0}g_{j+1}w,$$

holds for every function $f$ and every $s > 0$, with $C$ depending only on the $A_p$ constant for $w$. Now consider $G = \sum_{j=0}^{\infty} (C+1)^{-j}g_j$ with $C$ the same as above. Since $(C+1)^{-j}\|g_j\|_{(p/p_0)',w} \leq (C/(C+1))^j\|g\|_{(p/p_0)',w}$, the series converges in $L^{(p/p_0)'}(w)$ and we get $G \geq g$ with $\|G\|_{(p/p_0)',w} \leq (C+1)\|g\|_{(p/p_0)',w}$ and also, adding in $j$ the inequalities (3),

$$\int_{\{x : Mf(x) > s\}} Gw \leq C(C+1)s^{-p_0} \int_{\mathbb{R}^n} |f(x)|^{p_0}Gw,$$

for every $s > 0$; in other words: $Gw \in A_{p_0}$ and its $A_{p_0}$ constant depends only on the $A_p$ constant for $w$.

(b) Now $1 < p < p_0$ and $w \in A_p$. This is the same as saying that $1 < p_0' < p$ and $u \equiv w^{-1/(p-1)} \in A_p$. Therefore, we can apply part (a) to conclude that, for every $h \geq 0$ belonging to $L^{(p'/p_0)'}(u)$, there is an $H \geq h$ such that $\|H\|_{(p'/p_0)',u} \leq C\|h\|_{(p'/p_0)',u}$ and $H \in A_{p_0}$, with $C$ and the $A_{p_0}$ constant for $H$ depending only on the $A_p$ constant for $w$. But $(p'/p_0)' = (p_0 - 1)p/(p_0 - p)$, so that $h \in L^{(p'/p_0)'}(u)$ if and only if $h^{p_0-1}w^{-(p_0-p)/(p-1)} \in L^{(p_0-p)/(p-1)}(w)$. Also $H \in A_{p_0}$ if and only if $(Hu)^{-1/(p_0-p)} = [H^{p_0-1}w^{-(p_0-p)/(p-1)}]^{-1}w \in A_{p_0}$. Thus, if we are given $g \geq 0$ in $L^{(p_0-p)/(p-1)}(w)$, we just write $g = h^{p_0-1}w^{-(p_0-p)/(p-1)}$ with $h \in L^{(p'/p_0)'}(u)$, obtain the corresponding $H$ and then define $G = H^{p_0-1}w^{-(p_0-p)/(p-1)}$. This proves part (b).

We are ready to give the

PROOF OF THE THEOREM. (a) Let $1 \leq p_0 < p$, $w \in A_p$ and $f \in L^p(w)$,

$$\|Tf\|_{p_0,w} = \|Tf\|_{p/p_0,w} = \int_{\mathbb{R}^n} |Tf|^{p_0}gw$$

for some $g \geq 0$ with $\|g\|_{(p/p_0)',w} = 1$. Associate with $g$ a function $G$ as in Lemma 2. Then, the last integral is bounded by

$$\int_{\mathbb{R}^n} |Tf|^{p_0}Gw \leq C\int_{\mathbb{R}^n} |f|^{p_0}Gw \leq C\|f\|^{p_0}_{p/p_0,w}||G||_{(p/p_0)',w} \leq C\|f\|_{p_0,w}.$$  

(b) Let $1 < p < p_0$, $w \in A_p$ and $f \in L^p(w)$,

$$\|f\|_{p_0,w} = \|f\|_{p/p_0,w} = \int_{\mathbb{R}^n} |f|^{p_0}g^{-1}w$$

for some $g \geq 0$ such that $\|g\|_{p/(p_0-p),w} = 1$. Associate with $g$ a function $G$ as in part (b) of Lemma 2. Then

$$\|Tf\|_{p_0,w} = \|Tf\|_{p/p_0,w} \leq \left(\int_{\mathbb{R}^n} |Tf|^{p_0}G^{-1}w\right)\|G||_{p/(p_0-p),w} \leq C\int_{\mathbb{R}^n} |f|^{p_0}G^{-1}w \leq C\int_{\mathbb{R}^n} |f|^{p_0}g^{-1}w = C\|f\|_{p_0,w}.$$  

This ends the proof.  \(\square\)
OBSERVATION. There is a weak-type version of the theorem, in which one substitutes for (1) the inequality
\[(1') \int_{\{x : |Tf(x)| > s\}} w(x) \, dx \leq C s^{-p_o} \int_{\mathbb{R}^n} |f(x)|^{p_o} w(x) \, dx, \quad s > 0,\]
and also changes the inequality (2) in the conclusion to
\[(2') \int_{\{x : |Tf(x)| > s\}} w(x) \, dx \leq C s^{-p} \int_{\mathbb{R}^n} |f(x)|^{p} w(x) \, dx, \quad s > 0.\]

The proof of this weak-type version is as follows. For \(s > 0\), let us call \(E_s = \{x : |Tf(x)| > s\}\). For a set \(E\), \(w(E)\) will stand for \(\int_E w(x) \, dx\) and \(\chi_E\) will denote the characteristic function of \(E\).

(a) Let \(1 < p_o < p\), \(u; \in A_p\) and \(f \in L^p(w)\). Then, for \(s > 0\),
\[s^{p_o} w(E_s)^{p_o/p} = s^{p_o} \|\chi_{E_s}\|_{L^{p/p_o,w}} = s^{p_o} \int_{\mathbb{R}^n} \chi_{E_s} g w\]
for some \(g \geq 0\) with \(\|g\|_{(p/p_o)^*}^* = 1\). Associate with \(g\) a function \(G\) as in Lemma 2, part (a). Then
\[s^{p_o} w(E_s)^{p_o/p} \leq s^{p_o} \int_{\mathbb{R}^n} \chi_{E_s} G w \leq C \int_{\mathbb{R}^n} |G^p G w | \leq C \|f\|_{L^{p,w}}^p.\]
Thus, we have proved that \(w(E_s) \leq C s^{-p} \|f\|_{L^{p,w}}^p\), which is precisely (2').

(b) Let \(1 < p < p_o\), \(w \in A_p\) and \(f \in L^p(w)\). Choose \(g\) as in the proof of the corresponding part of the strong type theorem. Associate with \(g\) the function \(G\) as in Lemma 2 part (b). Then, for \(s > 0\),
\[s^{p_o} w(E_s)^{p_o/p} = s^{p_o} \|\chi_{E_s}\|_{L^{p/p_o,w}} \leq C s^{p_o} \int_{\mathbb{R}^n} \chi_{E_s} G^{-1} w \]
\[\leq C \int_{\mathbb{R}^n} |G^{-1} G^{-1} w | \leq C \int_{\mathbb{R}^n} |G^{-1} G^{-1} w | = C \|f\|_{L^{p,w}}^p,\]
which gives (2') as before.

The two theorems we have proved are also valid for the weights associated to other maximal operators. Some interesting examples are given in [7]. Our approach works for all of them because the description of the classes \(A_p\) is formally the same. These general weights do not necessarily satisfy a reverse Hölder's inequality. However, in the particular situation we are dealing with, the \(A_p\) weights do satisfy a reverse Hölder's inequality (see [1 and 5]). This fact allows us to strengthen the weak type theorem so that the strong type theorem becomes a corollary. Indeed, we can get the strong type inequalities (2) from the weak type inequalities (1'). We just need to realize that, because of the reverse Hölder's inequality, if \(w \in A_p\) and \(1 < p < \infty\), there is \(\epsilon > 0\) such that \(w \in A_{p-\epsilon}\) also. Then, Marcinkiewicz interpolation theorem applies, and we obtain the strong type estimate.

The case \(p = 1\) has been excluded. In fact, it has to be excluded. We can always extrapolate from \(p_0 = 1\) to any \(p\); but extrapolation down to \(p = 1\) is not legitimate. For example, the maximal partial sum operator of Fourier series is bounded in \(L^p(w)\) whenever \(1 < p < \infty\) and \(w \in A_p\) (see [3]), but it fails to be of weak type 1, 1 with respect to Lebesgue measure.

We should mention that the case \(t = 1\) of our Lemma 1 has been used by R. Coifman, P. Jones and J. L. Rubio de Francia [2] to give a simple constructive
proof of the factorization theorem of P. Jones [4]. Their proof simplifies a previous (nonconstructive) proof by J. L. Rubio de Francia [7], which used vector valued inequalities.

REFERENCES


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