ON EXTREME POINTS OF SUBORDINATION FAMILIES

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ABSTRACT. Let $F$ be the set of analytic functions in $U = \{ z: |z| < 1 \}$ subordinate to a univalent function $f$. Let $D = f(U)$. For $g(z) = f(\phi(z)) \in F$, let $\lambda(\theta)$ denote the distance between $g(e^{i\theta})$ and $\partial D$ (boundary of $D$). We obtain the following results.

1. If $f$ is Nevanlinna then $\int_0^{2\pi} \log(1 - |\phi(e^{i\theta})|) \, d\theta = -\infty$ if and only if

   \[ \int_0^{2\pi} \log(1 - |\phi(e^{i\theta})|) \, d\theta = -\infty. \]

2. If $g$ is an extreme point of the closed convex hull of $F$ then

   \[ \int_0^{2\pi} \log(1 - |\phi(e^{i\theta})|) \, d\theta = -\infty. \]

   for the case when $D$ is a Jordan domain subset to a half-plane and $f$ is Nevanlinna.

1. Introduction. Let $U = \{ z: |z| < 1 \}$ and let $A$ denote the set of functions analytic in $U$. Let $B_0$ denote the subset of $A$ consisting of functions $\phi$ that satisfy $|\phi(z)| < 1$ for $z \in U$ and $\phi(0) = 0$.

   Throughout this paper we assume that $f \in A$ and $f$ is univalent in $U$. Let $F$ denote the subset of $A$ consisting of functions $g$ that are subordinate to $f$ in $U$. This means that $g \in A$, $g(0) = f(0)$ and $g(U) \subset f(U)$. These conditions are equivalent to the existence of $\phi \in B_0$ so that $g(z) = f(\phi(z))$. $F$ is characterized by

   \[ g(z) = f(\phi(z)) \]

   where $\phi \in B_0$. Equation (1) defines a one-to-one correspondence between $F$ and $B_0$.

   Let $D$ denote $f(U)$. For $g \in F$, let

   \[ g(e^{i\theta}) = \lim_{r \to 1} g(re^{i\theta}). \]

   Since $f \in H^p$, for $p < \frac{1}{2}$, $g(e^{i\theta})$ exists almost everywhere. Let $\lambda(\theta)$ denote the distance between $g(e^{i\theta})$ and $\partial D$ where $\partial D$ denotes the boundary of $D$. T. H. MacGregor and the author [1] proved that if $f$ is convex, bounded, and if $\partial D$ is sufficiently smooth, then $g$ is an extreme point of $F$ if and only if

   \[ \int_0^{2\pi} \log \lambda(\theta) \, d\theta = -\infty. \]

   This result implies the well-known fact that $\phi$ is an extreme point of $B_0$ if and only if

   \[ \int_0^{2\pi} \log(1 - |\phi(e^{i\theta})|) \, d\theta = -\infty. \]

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It was also proved [1] that when \( f \) is bounded, convex and \( \partial D \) is sufficiently smooth the correspondence between \( F \) and \( B_0 \) given by \( g(z) = f(\phi(z)) \) provides a one-to-one correspondence between the extreme points of \( B_0 \) and the extreme points of \( F \). This is to say,

\[
\int_{0}^{2\pi} \log \lambda(\theta) \, d\theta = -\infty \quad \text{if and only if} \quad \int_{0}^{2\pi} \log \left(1 - |\phi(e^{i\theta})|\right) \, d\theta = -\infty.
\]

In \( \S 2 \), we prove that statement (5) holds for the case when \( f \) is univalent and \( f' \) is in the Nevanlinna class of analytic functions.

In \( \S 3 \), we give a necessary condition on the extreme points of the closed convex hull of \( F \) for the case when \( D = f(U) \) lies in a half-plane and \( \partial D \) is a Jordan curve.

2. Functions subordinate to a univalent function with a Nevanlinna derivative. We let \( d(z, \Gamma) \) denote the distance between \( z \) and a closed set \( \Gamma \), \( m(A) \) denote the Lebesgue measure of \( A \) and \( \log^+ x = \max\{0, \log x\} \).

**Theorem 1.** Let \( f \) be analytic and univalent in \( U \). Assume that \( f' \) is in the Nevanlinna class. Let \( D \) denote \( f(U) \) and let \( F \) denote the set of functions subordinate to \( f \). For \( g(z) = f(\phi(z)) \in F \), let \( \lambda(\theta) \) denote \( d(g(e^{i\theta}), \partial D) \). Then:

(a) \( \int_{0}^{2\pi} \log^+ \lambda(\theta) \, d\theta \) is convergent.

(b) \( \int_{0}^{2\pi} \lambda(\theta) \, d\theta = -\infty \) if and only if \( \int_{0}^{2\pi} \log \left(1 - |\phi(e^{i\theta})|\right) \, d\theta = -\infty \).

**Proof.** Since \( f \) is univalent, it follows that

\[
\int_{0}^{2\pi} \log^+ \lambda(\theta) \, d\theta \leq \int_{0}^{2\pi} \log \left(1 - |\phi(e^{i\theta})|\right) \, d\theta.
\]

When \( g(e^{i\theta}) \) and \( \phi(e^{i\theta}) \) exist and \( |\phi(e^{i\theta})| < 1 \), we obtain

\[
\int_{0}^{2\pi} \log \lambda(\theta) \, d\theta = -\infty \quad \text{if and only if} \quad \int_{0}^{2\pi} \log \left(1 - |\phi(e^{i\theta})|\right) \, d\theta = -\infty.
\]

Next, let \( A = \{\theta : g(e^{i\theta}) \text{ exists and } \lambda(\theta) = 0\} \). If \( m(A) > 0 \) then it follows that \( \int_{0}^{2\pi} \log \lambda(\theta) \, d\theta = \int_{0}^{2\pi} \log \left(1 - |\phi(e^{i\theta})|\right) \, d\theta = -\infty \). Assume for the rest of the proof that \( m(A) = 0 \). This implies that (7) holds for almost all \( \theta \). The fact that \( 1 \leq 1 + |\phi(e^{i\theta})| \leq 2 \) reduces (7) to

\[
\int_{0}^{2\pi} \log \lambda(\theta) \, d\theta \leq \int_{0}^{2\pi} \log \left(1 - |\phi(e^{i\theta})|\right) \, d\theta = -\infty.
\]

which also holds for almost every \( \theta \). Thus we conclude that \( -\infty \leq \int_{0}^{2\pi} \log \lambda(\theta) \, d\theta < M \), for some constant \( M \), because \( \log^+ \lambda(\theta) \in L^1 \). This together with (8) implies that (9)

\[
-2\pi \log 4 + \int_{0}^{2\pi} \log |f'(\phi(e^{i\theta}))| \, d\theta + \int_{0}^{2\pi} \log \left(1 - |\phi(e^{i\theta})|\right) \, d\theta \leq \int_{0}^{2\pi} \log \lambda(\theta) \, d\theta
\]

and so part (b) follows.
We close this section by noting that the conclusion of Theorem 1 is true for the case when \( g \) is subordinate to a close to convex function \( f \). This is so because it was shown that \( f' \in H^{1/3} \) [3] and thus \( f' \) is Nevanlinna.

3. Jordan domains. We let \( cA \) denote \( C \setminus A \).

**Lemma 1.** Let \( D \) be a bounded Jordan domain (\( \partial D \) is a Jordan curve). Let \( g \) be a nonconstant bounded analytic function in \( U \). If \( g(e^{i\theta}) \in \overline{D} \) for almost all \( \theta \) then \( g(U) \subseteq D \).

**Proof.** Let \( G = g(U) \). We want to show that \( G \subseteq D \). We shall show first that \( \partial G \subseteq D \). Assume, to the contrary, that there is a point \( w_0 \in \partial G \) and \( w_0 \notin \overline{D} \). Let \( \epsilon = d(w_0, D) \). Since \( w_0 \in \partial G \) and \( w_0 \notin \overline{D} \), there exists a point \( w_1 \in cG \) and \( w_1 \notin \overline{D} \) so that \( |w_0 - w_1| < \epsilon /2 \). It follows that \( d(w_1, D) > \epsilon /2 \). Let

\[
(10) \quad h(z) = \frac{1}{g(z) - w_1}, \quad z \in U,
\]

\( h(z) \) is analytic, bounded and \( h(e^{i\theta}) = 1/(g(e^{i\theta}) - w_1) \) for almost all \( \theta \). Since \( g(e^{i\theta}) \in \overline{D} \) for almost all \( \theta \), it follows that \( |h(e^{i\theta})| \leq 2/\epsilon \) for almost all \( \theta \). The Poisson Formula implies that \( |h(z)| \leq 2/\epsilon \) for every \( z \in U \). This contradicts \( |w_1 - w_0| < \epsilon /2 \). Hence \( \partial G \subseteq \overline{D} \).

Next, we shall show that \( L = \overline{G} \cap c\overline{D} \) is open. Let \( w \in L \). Since \( c\overline{D} \) is open, there is a neighborhood of \( w, N_\epsilon(w) \), so that \( N_\epsilon(w) \subseteq c\overline{D} \). \( N_{\epsilon/2}(w) \subseteq L \), because if not then \( N_{\epsilon/2}(w) \cap c\overline{G} \neq \emptyset \) and, since \( N_{\epsilon/2}(w) \cap \overline{G} \neq \emptyset \), one concludes that \( N_{\epsilon/2}(w) \cap \partial \overline{G} \neq \emptyset \). Thus there exists \( w_0 \in N_{\epsilon/2}(w) \cap \partial \overline{G} \) and \( w_0 \notin \overline{D} \). This then contradicts the first part of the proof of the lemma. Hence \( L \) is open.

Let \( u \in \partial L \) and assume that \( u \notin \overline{D} \). Since \( cL = c\overline{G} \cup \overline{D} \) it follows that every neighborhood of \( u \), with radius less than \( d(u, D) \), intersects \( \overline{G} \). This implies that \( u \in \partial \overline{D} \) and consequently \( u \in \overline{D} \). Hence \( \partial L \subseteq \partial D \) and consequently \( c\overline{D} = L \cup (cL \cap c\overline{D}) \). Since \( c\overline{D} \) is connected (\( \partial D \) is a Jordan curve) and since \( L \) is bounded, we conclude that \( L \) is empty and then \( G \subseteq \overline{D} \). This and Jordan’s Theorem [7, p. 115] imply that \( G \subseteq D \).

The following lemma is a generalization of Lemma 1.

**Lemma 2.** Let \( D \) be a Jordan domain subset to a half-plane \( H \). Let \( g \) be a nonconstant function analytic in \( U \) so that \( g(U) \subseteq H \). If \( g(e^{i\theta}) \in \overline{D} \) for almost every \( \theta \) then \( g(U) \subseteq D \).

**Proof.** Let \( T \) be a m"obius transformation that maps \( H \) onto \( U \). Let \( h(z) = T(g(z)) \). \( h(z) \) is bounded, analytic, \( h(e^{i\theta}) = T(g(e^{i\theta})) \) exists for almost all \( \theta \) and \( h(e^{i\theta}) \in \overline{T(D)} \). Since \( T \) is a homeomorphism and \( \partial \overline{D} \) is a Jordan curve, it follows that \( \partial(T(D)) \) is a Jordan curve. Hence, by Lemma 1, \( h(U) \subseteq T(D) \) and consequently \( g(U) \subseteq D \).

We now apply Lemma 2 to get the following theorem.

**Theorem 2.** Let \( f \) be a univalent analytic function in \( U \). Assume that \( D = f(U) \) is a Jordan domain subset to a half-plane \( H \). Let \( F \) be the set of analytic functions subordinate to \( f \). If \( g \) is an extreme point of the closed convex hull of \( F \) then \( \int_0^\pi \log \lambda(\theta)/(1 + \lambda(\theta)) \, d\theta = -\infty \).
Remark. \( \lambda(\theta)/(1 + \lambda(\theta)) \) can be replaced by \( \lambda(\theta) \) when \( f \) is bounded.

**Proof.** Assume that \( \int_0^{2\pi} \log(\lambda(\theta)/(1 + \lambda(\theta))) d\theta > -\infty \). Since \( \lambda(\theta)/(1 + \lambda(\theta)) < 1, \log(\lambda(\theta)/(1 + \lambda(\theta))) \in L^1 \). Let

\[
(11) \quad h(z) = z \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \frac{\lambda(t)}{1 + \lambda(t)} \, dt \right\}.
\]

It is known that \( h \in H^\infty \) and \( |h(e^{i\theta})| = \lambda(\theta)/(1 + \lambda(\theta)) \) for almost all \( \theta \) [2, pp. 24, 126]. Since \( |h(e^{i\theta})| \leq \lambda(\theta) \), it follows that \( g(e^{i\theta}) \pm h(e^{i\theta}) \in \overline{D} \) for almost all \( \theta \). Moreover, \( h \in H^\infty \) implies that \( g(z) \pm h(z) \) is in a half-plane \( H_1 \), containing \( H \), for all \( z \in U \). Thus, by Lemma 2, it follows that \( g(z) \pm h(z) \in D \) for all \( z \in U \) and so \( g(z) \pm h(z) \in F \). Since \( h \not\equiv 0 \), \( g \) cannot be an extreme point.

We come now to the main result of this section.

**Theorem 3.** Let \( f \) be a univalent analytic function in \( U \). Assume that \( f' \) is Nevanlinna and \( D = f(U) \) is a Jordan domain subset to a half-plane \( H \). Let \( F \) be the set of analytic functions subordinate to \( f \). If \( g(z) = f(\phi(z)) \) is an extreme point of the closed convex hull of \( F \) then \( \int_0^{2\pi} \log(1 - |\phi(e^{i\theta})|) \, d\theta = -\infty \).

**Remark.** In other words, \( \{g \in F: g \text{ is an extreme point of the closed convex hull of } F\} \subset \{f(\phi): \phi \text{ is an extreme point of } B_0\} \).

**Proof.** Theorem 2 implies that \( \int_0^{2\pi} \log(\lambda(\theta)/(1 + \lambda(\theta))) d\theta = -\infty \). Let \( E = \{\theta: \lambda(\theta) \text{ exists and } \lambda(\theta) < 1\} \) and let \( G = \{\theta: \lambda(\theta) \text{ exists and } \lambda(\theta) > 1\} \). \( m(E \cup G) = 2\pi \). For \( \theta \in E \), we have

\[
(12) \quad \frac{\lambda(\theta)}{2} < \frac{\lambda(\theta)}{1 + \lambda(\theta)} < \lambda(\theta)
\]

and for \( \theta \in G \), we have \( 1 + \lambda(\theta) < 2\lambda(\theta) \) and so

\[
(13) \quad \frac{1}{2} < \frac{\lambda(\theta)}{1 + \lambda(\theta)} < 1.
\]

(13) implies that \( \int_G \log(\lambda(\theta)/(1 + \lambda(\theta))) d\theta \) is convergent. Therefore,

\[
\int_0^{2\pi} \log \frac{\lambda(\theta)}{1 + \lambda(\theta)} d\theta = \int_E \log \frac{\lambda(\theta)}{1 + \lambda(\theta)} d\theta = -\infty
\]

and by (12) \( \int_E \log \lambda(\theta) d\theta = -\infty \). Because of Theorem 1 this gives that \( \int_0^{2\pi} \log \lambda(\theta) d\theta = -\infty \) and consequently \( \int_0^{2\pi} \log(1 - |\phi(e^{i\theta})|) d\theta = -\infty \).

**Remarks.** 1. The conclusion of Theorem 3 follows for the case when \( f \) is convex. This is because \( f(D) \) is a Jordan domain and \( f' \in H^{1/2} \) [3].

2. Theorem 2 was proved by T. H. MacGregor and the author [1] for the case when \( f \) is convex and \( f(U) \) is not a half-plane.

3. The converse of Theorem 3 does not hold in general. For example, the extreme points of \( F \), when \( f = (1 + z)/(1 - z) \), are characterized by

\[
g = \frac{1 + xz}{1 - xz}, \quad |x| = 1.
\]

Other examples in [1 and 6] show this claim.

4. We conjecture that Theorems 1 and 3 hold for any unrestricted univalent function \( f \).
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References


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