PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 87, Number 3, March 1983

A NOTE ON WEIGHTED NORM INEQUALITIES FOR THE HARDY-LITTLEWOOD MAXIMAL OPERATOR

MICHAEL CHRIST AND ROBERT FEFFERMAN

ABSTRACT. In this note we give an extremely simple proof of the weight norm inequalities for the Hardy-Littlewood maximal operator in $\mathbf{R}^n$.

The purpose of this note is to provide an extremely elementary proof of the boundedness of the maximal operator, $\|Mf\|_{L^p(w)} \leq C_p \|f\|_{L^p(w)}$ when the weight $w \in A_p$, $p > 1$. The result is well known and is due to Muckenhoupt [**1**]. The proof was simplified in R. Coifman and C. Fefferman [**2**], but still depended on a "reverse Hölder" inequality and was not totally elementary. Rather recently Sawyer in [**3**] solved the two weight problem for the maximal operator, which yielded a simple new proof in the one weight case, but which involved a condition on the weight not obviously equivalent to A_p. In [**4**] Hunt, Kurtz, and Neugebauer proved this equivalence and their proof avoids the reverse Hölder inequality. Finally, our proof avoids both the reverse Hölder inequality and also the reformulation of the A_p condition. The proof is very similar to Sawyer's [**3**] and Christ's alternate proof to Sawyer's [**5**].

We shall assume the reader is familiar with the basic definitions from weighted inequalities as appear in, say, [**2**]. Then, let $f \in L^p(w)$ where $w \in A_p(\mathbf{R}^n)$, $p > 1$. We Calderón-Zygmund decompose f at heights C_n^k for k an integer, where C_n is some large constant depending only on the dimension n, and which we specify later. Let Q_j^k, $j \geq 1$, be the Calderón-Zygmund cubes at height C_n^k. Let $E_j^k = Q_j^k - \bigcup_{Q_m^l \subset Q_j^k} Q_m^l$. Then we have

$$\int_{\mathbf{R}^n} (Mf)^p w \leq B \sum_{k,j} \left(\frac{1}{|Q_j^k|} \int_{Q_j^k} f \right)^p w(E_j^k)$$

(henceforth we write $q(E)$ for $\int_E q$ where q is in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$, $q_E = q(E)/|E|$, and following Sawyer, set $\sigma = w^{-1/(p-1)}$)

$$\leq B \sum_{k,j} \left[\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} (f\sigma^{-1})\sigma \right]^p \sigma(E_j^k) \cdot \left\{ \frac{w(E_j^k)}{\sigma(E_j^k)} \left(\frac{\sigma(Q_j^k)}{|Q_j^k|} \right)^p \right\}.$$

Received by the editors June 15, 1982.
1980 *Mathematics Subject Classification*. Primary 42B25.
Key words and phrases. Maximal functions, weights.

So far we have just used arithmetic. Now observe that if C_n is large enough $|E_j^k| > \frac{1}{2}|Q_j^k|$. Since $\sigma \in A_{p'}$, it is A_∞ and so $\sigma(E_j^k) \geqslant \eta(\sigma(Q_j^k))$ for some $\eta > 0$. (Indeed, assuming $\sigma_{Q_j^k} = 1$, then $\sigma(E_j^k)/\sigma(Q_j^k) \geqslant \frac{1}{2}\sigma_{E_j^k}$. But

$$\left(\sigma_{E_j^k}\right)^{-1/(p'-1)} \leqslant \left(\sigma^{-1/(p'-1)}\right)_{E_j^k} \leqslant 2\|\sigma\|_{A_{p'}}^{1/(p'-1)}.)$$

Taking this into account, the quantity in braces is bounded since $w \in A_p$. What is left is just dominated by

$$(*) \qquad \int_{\mathbf{R}^n} M_\sigma^p(f\sigma^{-1})\sigma$$

(where $M_\mu(f)(x) = \sup_{x \in Q}(1/\mu(Q))\int_Q |f|\, d\mu$ for a measure μ). But M_σ is bounded on $L^p(\sigma)$ and so $(*)$

$$\leqslant B'\int_{\mathbf{R}^n} f^p\sigma^{-p+1} = B'\int_{\mathbf{R}^n} f^p w.$$

Bibliography

1. B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226.
2. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. **51** (1974), 241–250.
3. E. Sawyer, *A characterization of a two weight norm inequality for maximal operators* (to appear).
4. R. Hunt, D. Kurtz and C. Neugebauer, *A note on the equivalence of A_p and Sawyer's condition for equal weights* (to appear).
5. M. Christ, *Weighted norm inequalities and Schur's lemma* (preprint).

Department of Mathematics, University of Chicago, Chicago, Illinois 60637 (Current address of Robert Fefferman)

Current address (Michael Christ): Department of Mathematics, Princeton University, Princeton, New Jersey 08544