A NOTE ON WEIGHTED NORM INEQUALITIES FOR THE HARDY-LITTLEWOOD MAXIMAL OPERATOR

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ABSTRACT. In this note we give an extremely simple proof of the weight norm inequalities for the Hardy-Littlewood maximal operator in $\mathbb{R}^n$.

The purpose of this note is to provide an extremely elementary proof of the boundedness of the maximal operator, $\|Mf\|_{L^p(w)} \leq C_p \|f\|_{L^p(w)}$ when the weight $w \in A_p$, $p > 1$. The result is well known and is due to Muckenhoupt [1]. The proof was simplified in R. Coifman and C. Fefferman [2], but still depended on a “reverse Hölder” inequality and was not totally elementary. Rather recently Sawyer in [3] solved the two weight problem for the maximal operator, which yielded a simple new proof in the one weight case, but which involved a condition on the weight not obviously equivalent to $A_p$. In [4] Hunt, Kurtz, and Neugebauer proved this equivalence and their proof avoids the reverse Hölder inequality. Finally, our proof avoids both the reverse Hölder inequality and also the reformulation of the $A_p$ condition. The proof is very similar to Sawyer’s [3] and Christ’s alternate proof to Sawyer’s [5].

We shall assume the reader is familiar with the basic definitions from weighted inequalities as appear in, say, [2]. Then, let $f \in L^p(w)$ where $w \in A_p(\mathbb{R}^n)$, $p > 1$. We Calderón-Zygmund decompose $f$ at heights $C_n^k$ for $k$ an integer, where $C_n$ is some large constant depending only on the dimension $n$, and which we specify later. Let $Q_j^k$, $j \geq 1$, be the Calderón-Zygmund cubes at height $C_n^k$. Let $E_j^k = Q_j^k \setminus \bigcup_{Q_m^k \subset Q_j^k} Q_m^k$. Then we have

$$\int_{\mathbb{R}^n} (Mf)^p w \leq B \sum_{k,j} \left( \frac{1}{|Q_j^k|} \int_{Q_j^k} f \right)^p w(E_j^k)$$

(henceforth we write $q(E)$ for $\int_E q$ where $q$ is in $L^1_{\text{loc}}(\mathbb{R}^n)$, $q_E = q(E)/|E|$, and following Sawyer, set $\sigma = w^{-1/(p-1)}$)

$$\leq B \sum_{k,j} \left[ \frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} (fs^{-1}) \right]^p \sigma(E_j^k) \left\{ \frac{w(E_j^k)}{\sigma(E_j^k)} \left( \frac{\sigma(Q_j^k)}{|Q_j^k|} \right)^p \right\}.$$
So far we have just used arithmetic. Now observe that if $C_\sigma$ is large enough $|E_\sigma^k| > \frac{1}{2} |Q_k^k|$. Since $\sigma \in A_\rho$, it is $A_\infty$ and so $\sigma(E_\sigma^k) \geq \eta(\sigma(Q_k^k))$ for some $\eta > 0$. (Indeed, assuming $\sigma(Q_k^k) = 1$, then $\sigma(E_\sigma^k)/\sigma(Q_k^k) \geq \frac{1}{2} \sigma(E_\sigma^k)$. But

$$(\sigma(E_\sigma^k))^{-1/(\rho-1)} \leq (\sigma^{-1/(\rho-1)}(E_\sigma^k) \leq 2 \|\sigma\|_{A_\rho}^{-1/(\rho-1)}).$$

Taking this into account, the quantity in braces is bounded since $w \in A_\rho$. What is left is just dominated by

$$(\ast) \int_{\mathbb{R}^n} M_\sigma^\rho (f^\sigma) d\sigma$$

(where $M_\mu(f)(x) = \sup_{x \in Q} (1/\mu(Q)) \int_Q |f| d\mu$ for a measure $\mu$). But $M_\sigma$ is bounded on $L^p(\sigma)$ and so $(\ast)$

$$\leq B' \int_{\mathbb{R}^n} f^\sigma \sigma^{\rho-1} = B' \int_{\mathbb{R}^n} f^\rho w.$$

**Bibliography**


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