PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 87, Number 3, March 1983

# ON THE ORDER AND LOWER ORDER OF ENTIRE FUNCTIONS WITH RADIALLY DISTRIBUTED ZEROS

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ABSTRACT. It is shown that the order and lower order of an entire function with zeros restricted to $k$ distinct rays differ at most by $k$, if either $k \leq 2$ or if the zeros or the rays are regularly distributed.

**1. Introduction.** Throughout this paper $f$ will denote an entire function whose zeros are restricted to $k$ distinct rays $\arg z = \omega_j$ $(0 \leq \omega_1 < \omega_2 < \cdots < \omega_k < 2\pi)$. The order and lower order of $f$ are defined to be

$$\begin{matrix}\lambda(f) \\ \mu(f)\end{matrix} = \lim \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log T(r,f)}{\log r} \qquad (r \to \infty).$$

By $N(r)$ and $N_j(r)$ we denote the integrated counting functions of all zeros of $f$ and of those on $\arg z = \omega_j$, respectively.

It was proved by Edrei and Fuchs **[2]** (see also **[1, 3, 4, 6]**) that the order and the lower order are cofinite. But, in general, no explicit upper bound for $\lambda(f)$ is known. Edrei and Fuchs **[2]** constructed an entire function $F$ with real negative zeros of prescribed order $\lambda$ and lower order $\mu$, subject only to $0 < \mu \leq \lambda < 1$. Therefore, the least upper bound for $\lambda(f)$ in terms of $\mu(f)$ and $k$, if there is any, is at least $[\mu(f)] + k$, where [ ] denotes the greatest integer function (consider $f(z) = F(z^k)$, $k$ a positive integer). The inequality

$$\lambda(f) \leq [\mu(f)] + k \tag{1}$$

is known to be true for $k = 1$ (see **[1, 3, 6]**). It will be shown that (1) is valid for $k = 2$, too, and also, for general $k$, if the rays $\arg z = \omega_j$ or the zeros of $f$ are regularly distributed in some sense.

**2. Statement of results.** There is no loss of generality in assuming $f(0) = 1$. If $f$ has finite lower order, let $q$ be the smallest integer such that

$$\liminf_{r \to \infty} \frac{T(r,f)}{r^{q+1}} < \infty.$$

Clearly, $q \leq \mu(f) \leq q + 1$.

Received by the editors June 10, 1982.
1980 *Mathematics Subject Classification*. Primary 30D15; Secondary 30D20.
*Key words and phrases*. Order, lower order of an entire function, distribution of zeros.


0002-9939/82/0000-0927/$01.75

All our results will be derived from the following

THEOREM. *Under the hypotheses stated above there exists a sequence* $r_n \uparrow \infty$ *such that, for any integer* $p$, $p > q$,

$$\sum_{j=1}^{k} e^{ip\omega_j} \int_0^{r_n} \frac{N_j(t)}{t^{p+1}}\, dt = O(1) \tag{2}$$

*as* $n \to \infty$.

COROLLARY 1. $\mu(f) < \infty$ *implies* $\lambda(f) < \infty$.

COROLLARY 2. $k = 1$ *and* $\mu(f) < \infty$ *imply* $\lambda(f) \leqslant q + 1 \leqslant [\mu(f)] + 1$.

COROLLARY 3. $k = 2$ *and* $\mu(f) < \infty$ *imply either* $\lambda(f) \leqslant q + 1$ *or* $q + 1 < \lambda(f) \leqslant q + 2 \leqslant [\mu(f)] + 2$. *In this case,* $\omega_2 - \omega_1 = (2s + 1)\pi/(q + 1)$ ($s, 0 \leqslant s \leqslant q$, *an integer*).

REMARK. In [1] Abi-Khuzam proved for $k = 2$, $\omega_1 = 0$, $\omega_2 = m\pi/\alpha$ ($m$ and $\alpha$ relative prime positive integers): $\lambda(f) \leqslant [\mu(f)] + \alpha$ if $m$ is even, and $\lambda(f) \leqslant [\mu(f)] + 2\alpha$ if $m$ is odd. As Corollary 3 shows, this is only sharp if $\alpha = m = 1$, i.e. if $f$ has only real zeros (see also Volkmann [6]).

COROLLARY 4. *If* $\omega_j - \omega_{j-1} = 2\pi/k$ ($1 \leqslant j \leqslant k$; $\omega_0 = \omega_k - 2\pi$), *then* $\mu(f) < \infty$ *implies* $\lambda(f) \leqslant q + l \leqslant [\mu(f)] + k$, *where* $l$, $1 \leqslant l \leqslant k$, *is the smallest integer satisfying* $q + l = ks$ ($s$ *an integer*).

COROLLARY 5. *If the limits* $\alpha_j := \lim_{r\to\infty} N_j(r)/N(r)$ *exist, then* $\lambda(f) \leqslant q + k \leqslant [\mu(f)] + k$.

REMARK. Given $0 < \alpha \leqslant \beta < 1$, Edrei and Fuchs [2] constructed an entire function $F$ of lower order $\alpha$ and order $\beta$, having only negative zeros. Thus, $f(z) = F(-z^k)$ has lower order $\mu = k\alpha$ and order $\lambda = k\beta$, which may be chosen arbitrarily near to $[\mu] + k$. The zeros of $f$ are regularly distributed on $k$ distinct rays $\arg z = 2\pi j/k$ ($0 \leqslant j < k$), and so this example does not only prove that inequality (1) is sharp for every $k$ (if it is true in general), but also the sharpness of the statements of Corollaries 2–5.

**3. Proof of the Theorem.** Let $a_1, a_2, \ldots$ be the zeros of $f$ repeated according to multiplicity and assume $\log f(z) = c_1 z + c_2 z^2 + \cdots$ near $z = 0$. Then for any integer $p = 0, 1, 2, \ldots$

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\, e^{-ip\theta}\, d\theta = \frac{c_p}{2} r^p + \frac{1}{2p} \sum_{|a_\nu| \leqslant r} \left( \left(\frac{r}{a_\nu}\right)^p - \left(\frac{\bar{a}_\nu}{r}\right)^p \right) \tag{3}$$

(see F. Nevanlinna [5]). Clearly,

$$\left| \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})| e^{-ip\theta}\, d\theta \right| \leqslant 2T(r, f) \tag{4}$$

and

$$\left| \sum_{|a_\nu| \leq r} \left( \frac{\bar{a}_\nu}{r} \right)^p \right| \leq n(r) \leq T(er, f). \tag{5}$$

With obvious notation,

$$\sum_{\substack{|a_\nu| \leq r \\ \arg a_\nu = \omega_j}} a_\nu^{-p} = e^{-ip\omega_j} \int_0^r \frac{dn_j(t)}{t^p}. \tag{6}$$

Integrating by parts twice we get

$$\int_0^r \frac{dn_j(t)}{t^p} = \frac{n_j(r)}{r^p} + p\frac{N_j(r)}{r^p} + p^2 \int_0^r \frac{N_j(t)}{t^{p+1}}\, dt,$$

which yields together with (3)–(6), after passing to conjugate values,

$$\sum_{j=1}^{k} e^{ip\omega_j} \int_0^r \frac{N_j(t)}{t^{p+1}}\, dt = O\left( \frac{T(er, f)}{r^p} + 1 \right) \quad \text{as } r \to \infty. \tag{7}$$

To complete the proof, we have only to choose $r_n \uparrow \infty$ such that $T(er_n, f) r_n^{-q-1}$ remains bounded as $n \to \infty$.

**4. Proof of the corollaries.** In all cases we may replace $\lambda(f)$ by $\lambda(0, f) := \limsup_{r \to \infty} \log N(r)/\log r$, since $\lambda(0, f) < \lambda(f)$ implies $\lambda(f) = \mu(f)$. This is easily seen from $f = Pe^g$, where $P$ is a canonical product of order $\lambda(0, f) < \lambda(f)$ (or a polynomial). Since $e^g$ is of regular growth, the same is true for $f$. The method gives slightly more. Instead of inequalities of type $\lambda(0, f) \leq b$ $(= q + 1, q + 2, q + l$ and $q + k$, respectively) we will prove that $\int_0^\infty N(t)/t^{b+1}\, dt$ converges (implying $N(r)/r^b \to 0$ as $r \to \infty$).

Proof of Corollary 1. Assume that $\int_0^\infty N(t)/t^{p+1}\, dt$ diverges for some $p > q$. We divide equation (2) by $\int_0^{r_n} N(t)/t^{p+1}\, dt$ and choose a subsequence of $r_n$ (still denoted by $r_n$) such that the limits

$$\lim_{n \to \infty} \int_0^{r_n} \frac{N_j(t)}{t^{p+1}}\, dt \Big/ \int_0^{r_n} \frac{N(t)}{t^{p+1}}\, dt =: \alpha_j$$

exist. Then we get $\sum_{j=1}^k \alpha_j e^{ip\omega_j} = 0$ $(\alpha_j \geq 0, \sum_{j=1}^k \alpha_j = 1)$, which shows that the origin belongs to the convex hull of $\{e^{ip\omega_j}\colon 1 \leq j \leq k\}$ for any such $p$. But this is not possible for every $p$, since by Weyl's equidistribution theorem [7] there exist arbitrarily large $p$ such that all $e^{ip\omega_j}$, $1 \leq j \leq k$, belong to an open halfplane. This proves Corollary 1.

Proof of Corollary 2. We mention only that (2) implies the boundedness of $\int_0^r N(t)/t^{q+2}\, dt$ $(N_1(t) \equiv N(t))$ for $r = r_n$ and so, by monotonicity, for any $r \geq 0$.

Proof of Corollary 3. If $\int_0^\infty N(t)/t^{q+2}\, dt$ converges, we are done. If not, both integrals $\int_0^\infty N_j(t)/t^{q+2}\, dt$, $j = 1, 2$, must diverge. Thus, dividing by $\int_0^{r_n} N_2(t)/t^{q+2}\, dt$ and letting $n$ tend to infinity, (2) gives $-e^{i(q+1)(\omega_2 - \omega_1)} = 1$ and so $\omega_2 - \omega_1 = (2s + 1)\pi/(q + 1)$, $s$ an integer. In the same way, $\int_0^\infty N(t)/t^{q+3}\, dt = +\infty$ would imply $\omega_2 - \omega_1 = (2s' + 1)\pi/(q + 2)$, which is impossible.

PROOF OF COROLLARY 4. We may assume $\omega_j = 2(j-1)\pi/k$ $(1 \leqslant j \leqslant k)$. If $l, 1 \leqslant l \leqslant k$, is chosen in such a way that $q + l = ks$, then for $p = q + l$ we have $e^{ip\omega_l} = 1$ and (2) gives $\int_0^{r_n} N(t)/t^{q+l+1}\,dt = O(1)$, which proves Corollary 4.

PROOF OF COROLLARY 5. We assume that $\int_0^\infty N(t)/t^{p+1}\,dt$ diverges for any integer $p, q < p \leqslant q + k$, and will derive a contradiction. By l'Hospital's rule we get

$$\int_0^{r_n} \frac{N_j(t)}{t^{p+1}}\,dt \Big/ \int_0^{r_n} \frac{N(t)}{t^{p+1}}\,dt \to \alpha_j \quad \text{as } n \to \infty$$

(even if $\alpha_j$ is zero). Thus, (2) gives

$$\sum_{j=1}^{k} \alpha_j e^{ip\omega_j} = 0 \qquad (p = q+1, \ldots, q+k) \tag{8}$$

which is impossible, since (8) has no nontrivial solution (the determinant $| e^{ip\omega_j} |_{j=1,\ldots,k;\, p=q+1,\ldots,q+k}$ does not vanish). This proves Corollary 5.

REMARK. Obviously, Corollary 5 holds true under the following weaker hypothesis: For every $p, q < p \leqslant q + k$, there exists a subsequence of $r_n$ (still denoted by $r_n$) such that

$$\int_0^{r_n} \frac{N_j(t)}{t^{p+1}}\,dt \Big/ \int_0^{r_n} \frac{N(t)}{t^{p+1}}\,dt$$

tends to $\alpha_j$ (independent of $p$) as $n \to \infty$.

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