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ESSENTIAL SPECTRA OF OPERATORS IN THE CLASS $\mathfrak{B}_n(\Omega)$[1]

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ABSTRACT. For a connected open subset Ω of the plane and n a positive integer, let $\mathfrak{B}_n(\Omega)$ be the space introduced by Cowen and Douglas in their paper *Complex Geometry and Operator Theory*. Our paper deals with characterizing the essential spectrum of an operator T in $\mathfrak{B}_n(\Omega)$ for which $\sigma(T) = \overline{\Omega}$ and the point spectrum of T^* is empty. This class of operators forms an important part of $\mathfrak{B}_n(\Omega)$ denoted by $\mathfrak{B}_n'(\Omega)$. We use this characterization to give another proof of the result of Axler, Conway and McDonald on determining the essential spectrum of the Bergman operator.

Let $A_n(G) = \{S\colon T = S^* \text{ is in } \mathfrak{B}_n'(G^*)\}$. We also characterize the weighted shifts in $A_1(G)$.

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NOTATION AND TERMINOLOGY. The following notation will be retained throughout the rest of this paper:

$$\square = \text{empty set}, \qquad \mathbf{C} = \text{complex plane}.$$

For a subset E of the plane, ∂E is the boundary of E, int E (or E°) is the interior of E, and $\overline{E}$ is the closure of E.

$\mathcal{L}(\mathcal{H})$ will denote the collection of bounded linear operators on the complex Hilbert space $\mathcal{H}$.

For T in $\mathcal{L}(\mathcal{H})$, the spectrum of T, point spectrum of T, and approximate point spectrum of T will be denoted by $\sigma(T)$, $\sigma_p(T)$, and $\sigma_{ap}(T)$ respectively. See **[4]** for the appropriate definitions.

An operator T in $\mathcal{L}(\mathcal{H})$ is called a Fredholm operator if ran T is closed and both ker T and ker T^* are finite dimensional. The essential spectrum of T, denoted by $\sigma_e(T)$, is the set of all λ in $\mathbf{C}$ such that $T - \lambda$ is not Fredholm.

The closed linear span of a family of subspaces $\{\mathcal{H}_\alpha\}$ is $\bigvee_\alpha \mathcal{H}_\alpha$.

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If B is a subset of the complex plane, then $B^* = \{z: \bar{z} \in B\}$. Finally $B(z, \delta) = \{\lambda \in \mathbf{C}: |z - \lambda| < \delta\}$.

1. Essential spectrum. For a connected open subset Ω of the plane and n a positive integer, let $\mathcal{B}_n(\Omega)$ denote the operators T defined on the Hilbert space $\mathcal{H}$ which satisfy

(a) $\Omega \subseteq \sigma(T) = \{\omega \in \mathbf{C}: T - \omega$ is not invertible$\}$,

(b) $\operatorname{ran}(T - \omega) = \mathcal{H}$ for ω in Ω,

(c) $V_{\omega \in \Omega} \ker(T - \omega) = \mathcal{H}$, and

(d) $\dim \ker(T - \omega) = n$ for ω in Ω.

The space $\mathcal{B}_n(\Omega)$ has been introduced and investigated by Cowen and Douglas **[3]**.

In this section we want to determine the essential spectrum of operators in $\mathcal{B}'_n(\Omega)$, where $\mathcal{B}'_n(\Omega) = \{T \in \mathcal{B}_n(\Omega): \sigma(T) = \bar{\Omega}$ and $\sigma_p(T^*) = \square\}$, and, as a consequence, we deduce the result of Axler, Conway and McDonald **[1]** on characterizing the essential spectrum of the Bergman operator.

(1.1) THEOREM. *Let Ω be an open connected subset of the plane, let n be a positive integer, and let $T \in \mathcal{L}(\mathcal{H})$ such that*

(i) $\sigma(T) = \bar{\Omega}$,

(ii) $\operatorname{ran}(T - \omega)$ *is closed*, $\omega \in \Omega$,

(iii) $\dim \ker(T - \omega) = n$, $\omega \in \Omega$,

(iv) $\sigma_p(T^*) = \square$.

Let $\Omega_0 =$ all points $\omega_0 \in \bar{\Omega}$ such that $\operatorname{ran}(T - \omega_0)$ *is closed and there exist holomorphic $\mathcal{H}$-valued functions $\{e_i(\omega)\}_{i=1}^n$ defined on some neighborhood B of ω_0 such that for each ω in B $\{e_i(\omega)\}_{i=1}^n$ forms a basis for* $\ker(T - \omega)$. *Then $\sigma_e(T) = \partial\Omega_0$.*

PROOF. Let $\Omega' = \{\omega_0 \in \bar{\Omega}: \operatorname{ran}(T - \omega_0)$ is closed and $\dim \ker(T - \omega) = n$ for ω in some neighborhood B of $\omega_0\}$. Note that Ω' is open and $\Omega \subseteq \Omega' \subseteq \bar{\Omega}$. Hence the closure of Ω' is $\bar{\Omega}$. Also for $\omega \in \Omega'$, $T - \omega$ is Fredholm and $\operatorname{ind}(T - \omega) = n$. We show that $\sigma_e(T) = \partial\Omega'$ and $\Omega' = \Omega_0$.

If $\omega \notin \partial\Omega'$, either ω is not in the closure of Ω' (hence $\omega \notin \bar{\Omega}$), in which case $T - \omega$ is invertible and hence Fredholm, or ω is in the closure of Ω', in which case $\omega \in \Omega'$ and $T - \omega$ is Fredholm. So $\sigma_e(T) \subseteq \partial\Omega'$. If $\omega_0 \notin \sigma_e(T)$, then $T - \omega_0$ is Fredholm; thus $T - \omega$ is Fredholm for ω in some neighborhood B of ω_0 and $\operatorname{ind}(T - \omega)$ is constant on this neighborhood. If $B \cap \Omega' = \square$, then $\omega_0 \notin \partial\Omega'$. If $B \cap \Omega' \neq \square$, then $\operatorname{ind}(T - \omega) = n$, $\omega \in B$. Because $\sigma_p(T^*) = \square$, $\dim \ker(T - \omega) = n$, $\omega \in B$. Thus $B \subseteq \Omega'$, so $\omega_0 \notin \partial\Omega'$. It follows that $\sigma_e(T) = \partial\Omega'$.

The inclusion $\Omega' \subseteq \Omega_0$ follows from Šubin's Theorem (**[3]**, Cowen and Douglas). Also, it is easy to see that $\Omega_0 \subseteq \Omega'$. Hence $\Omega_0 = \Omega'$ and $\sigma_e(T) = \partial\Omega_0$. Q.E.D.

We now have proved the following.

(1.2) COROLLARY. *Let $T \in \mathcal{B}'_n(\Omega)$ and let Ω_0 be as in Theorem* 1.1. *Then $\sigma_e(T) = \partial\Omega_0$.*

For the rest of our discussion let G be a bounded, open, connected, nonempty subset of the complex plane $\mathbf{C}$ and let $L^2(G)$ denote the L^2-space of (Lebesgue) area measure restricted to G. Denote by $L^2_a(G)$ the subspace of analytic functions belonging to $L^2(G)$. The Bergman space $L_a^2(G)$ is actually a closed subspace of $L^2(G)$ and thus is a Hilbert space. Let T_z be the Bergman operator on $L_a^2(G)$ defined by $T_z f = zf$.

(1.3) DEFINITION. A point $\lambda \in \partial G$ is said to be removable with respect to $L_a^2(G)$ if there exists an open neighborhood V of λ such that every function in $L_a^2(G)$ can be extended to an analytic function on $G \cup V$. The set of all points on ∂G which are removable with respect to $L_a^2(G)$ is denoted by $\partial_{2-r}G$. The Bergman essential boundary of G, denoted by $\partial_{2-e}G$, is the set of all points of ∂G which are not removable with respect to $L_a^2(G)$; so $\partial_{2-e}G = \partial G \sim \partial_{2-r}G$.

It is shown in [**1**] that $G \cup \partial_{2-r}G$ is an open subset of $\mathbf{C}$ and that $\partial_{2-r}G$ has zero area. Therefore it makes sense to consider $L_a^2(G \cup \partial_{2-r}G)$ and it is easy to see that $L_a^2(G) = L_a^2(G \cup \partial_{2-r}G)$, where the equality means that there is an obvious isometry between the two spaces. For $\lambda \in G \cup \partial_{2-r}G$, the linear functional on

$$L_a^2(G \cup \partial_{2-r}G) = L_a^2(G)$$

which takes h to $h(\lambda)$ is bounded (see [**2**, p. 5]).

(1.4) THEOREM. *Let* $G_0 = \{\lambda_0 \in \bar{G}\colon \operatorname{ran}(T_z - \lambda_0)$ *is closed and there exists a conjugate holomorphic* $L_a^2(G)$*-valued function* $d(\lambda)$ *defined on some neighborhood* V *of* λ_0 *such that* $\ker(T_z^* - \bar{\lambda}) = [d(\lambda)]$ *for* $\lambda \in V\}$. *Then* $G_0 = G \cup \partial_{2-r}G$. *Equivalently* $\sigma_e(T_z) = \partial_{2-e}G$.

PROOF. First suppose that $\lambda \in G \cup \partial_{2-r}G$. Then it is easy to verify that the range of $T_z - \lambda$ is equal to the kernel of the linear functional on $L_a^2(G) = L_a^2(G \cup \partial_{2-r}G)$ which sends h to $h(\lambda)$. In particular, the range of $T_z - \lambda$ is a closed subspace of $L_a^2(G)$ of codimension 1. Since $\ker(T_z - \lambda) = (0)$, we conclude that $T_z - \lambda$ is Fredholm (with index -1) and so $\lambda \notin \sigma_e(T_z)$. Thus $G \cup \partial_{2-r}G \subseteq \bar{G} \sim \sigma_e(T_z)$.

Next choose $\lambda_0 \in G_0$. Let $h(\lambda) = (1, d(\lambda))$. If $h(\lambda) \equiv 0$ for λ in a neighborhood of λ_0, then since $\lambda_0 \in \bar{G}$, there exists λ_1 in G such that $h(\lambda_1) = 0$. This shows that $1 \in \operatorname{ran}(T_z - \lambda_1)$, or equivalently $(z - \lambda_1)^{-1}$ is in $L_a^2(G)$ and this is absurd. Therefore if $\lambda_0 \in G_0$, then h is holomorphic in a neighborhood of λ_0 and does not vanish identically there. Hence there exists $r > 0$ such that $B(\lambda_0, r) \subseteq G_0$ and for λ in $B(\lambda_0, r)$, h is holomorphic and $h(\lambda) \neq 0$ for λ in $B_0 = B(\lambda_0, r) \sim \{\lambda_0\}$.

Now for λ in B_0 define $k(\lambda) = \overline{h(\lambda)}^{-1} d(\lambda)$. Obviously $k(\lambda)$, $\lambda \in B_0$ has the same properties as $d(\lambda)$. Furthermore it is normalized such that $(1, k(\lambda)) = 1$, $\lambda \in B_0$. Let $\{e(\lambda)\}$ be the Bergman kernel (reproducing kernel) for $L_a^2(G)$. Then for $\lambda \in G \cap B_0$, $\ker(T_z^* - \bar{\lambda}) = [e(\lambda)] = [k(\lambda)]$. Hence $e(\lambda) = \overline{c(\lambda)} k(\lambda)$, where $c(\lambda) \in \mathbf{C}$. By normalization, $1 = (1, e(\lambda)) = c(\lambda)(1, k(\lambda)) = c(\lambda)$ and therefore $e(\lambda) = k(\lambda)$. If $f \in L_a^2(G)$, then $f(\lambda) = (f, e(\lambda))$, $\lambda \in G$. Now define $\hat{f}(\lambda) = (f, k(\lambda))$, $\lambda \in B_0$. Clearly $\hat{f}(\lambda) = f(\lambda)$, $\lambda \in B_0 \cap G$. Hence $B_0 \subseteq G \cup \partial_{2-r}G$. If $f \in L_a^2(G) = L_a^2(G \cup \partial_{2-r}G)$, then $f \mid B_0$ is in $L_a^2(B_0)$ because $\partial_{2-r}G$ has zero area [**1**]. However, a function in $L_a^2(B_0)$ extends to B [**1**]. Thus each function in $L_a^2(G)$ extends to be analytic on B. This shows that $\lambda_0 \notin \partial_{2-e}G$. Hence $G_0 \subseteq G \cup \partial_{2-r}G$.

We will show that $G_0 = \bar{G} \sim \sigma_e(T_z)$. To see this let $\Omega = G^*$ and $T = T_z^*$. Then T satisfies the hypothesis of Theorem 1.1 with $n = 1$. Therefore $\sigma_e(T) = \partial\Omega_0$. It is easy to see that $\Omega_0 = G_0^*$. Hence $\sigma_e(T_z) = \sigma_e(T^*) = \sigma_e(T)^* = (\partial\Omega_0)^* = \partial\Omega_0^* = \partial G_0$. Since G_0 is open, $\overline{G} \sim \sigma_e(T_z) = \overline{G}_0 \sim \partial G_0 = G_0$.

We have already shown that $G_0 \subseteq G \cup \partial_{2-r}G \subseteq \overline{G} \sim \sigma_e(T_z)$; therefore $G_0 = G \cup \partial_{2-r}G = \overline{G} \sim \sigma_e(T)$. We conclude that $\sigma_e(T_z) = \partial_{2-e}G$. Q.E.D.

2. Weighted shifts in $A_1(G)$. For a connected open subset G of the plane and n a positive integer, let $A_n(G)$ denote the operators S in $\mathcal{L}(\mathcal{H})$ which satisfy

(a) $G \subseteq \sigma_p(S^*)^*$,

(b) $G \cap \sigma_{ap}(S) = \square$,

(c) $\sigma_p(S) = \square$,

(d) $\bigcap_{\lambda\in G}\operatorname{ran}(S-\lambda) = (0)$,

(e) $\dim\ker(S^* - \bar{\lambda}) = n$, $\lambda \in G$, and

(f) $\sigma(S) = \overline{G}$.

The space $A_n(G)$ is closely connected with the space $\mathcal{B}_n(\Omega)$. In particular it is easy to see that $A_n(G) = \{S: T = S^* \in \mathcal{B}_n'(G^*)\}$.

A *unilateral (bilateral) weighted shift* on a Hilbert space $\mathcal{H}$ is a linear map T from $\mathcal{H}$ to itself given by $Te_i = w_i e_{i+1}$, where $(e_i) = (e_i)_{i=0}^{\infty}$ $((e_i)_{i=-\infty}^{\infty})$ is an orthonormal basis for $\mathcal{H}$ and $\sup_k |w_k| < \infty$.

In his survey article, A. L. Shields [**5**] has investigated the properties of weighted shifts which we need in order to characterize the weighted shifts in $A_1(G)$. For a more detailed treatment of the subject and pertinent terminology see [**5**]. We now make a few observations.

Let S be in $A_n(G)$ and note that $\sigma(S) \sim \sigma_{ap}(S) = \{\partial\sigma(S) \sim \sigma_{ap}(S)\} \cup \{[\sigma(S)]^\circ \sim \sigma_{ap}(S)\}$. Because $\partial\sigma(S) \subseteq \sigma_{ap}(S)$ and $\sigma_{ap}(S)$ is closed (see Halmos [**4**, problem 62]), we conclude that $\sigma(S) \sim \sigma_{ap}(S)$ is an open set. Put $G_S = \sigma(S) \sim \sigma_{ap}(S)$; then $G \subseteq G_S \subseteq \sigma(S)$, so $\sigma(S) = \overline{G} = \overline{G}_S$. If $\lambda \in G_S$, then $\lambda \notin \sigma_{ap}(S)$, so $S - \lambda$ is bounded below. Therefore $S - \lambda$ has closed range. Also $\ker(S - \lambda) = (0)$, from which we conclude that $S - \lambda$ is semi-Fredholm. Now choose a sequence $\{\lambda_k\}$ in G such that $\lambda_k \to \lambda$; then $\operatorname{ind}(S - \lambda_k) \to \operatorname{ind}(S - \lambda)$. Now for S in $A_n(G)$ and $z \in G$, $S - z$ is a Fredholm operator and $\operatorname{ind}(S - z) = -n$. Therefore $\operatorname{ind}(S - \lambda) = -n$ and we conclude that $S \in A_n(G_S)$. Clearly G_S is the largest open subset G of the plane such that $S \in A_n(G)$.

If A is any operator let $m(A) = \inf\{\|Af\| : \|f\| = 1\}$. Then the following is true (see Shields [**5**]):

$$\sup_n [m(A^n)]^{1/n} = \lim_{n\to\infty} [m(A^n)]^{1/n}.$$

Let $r_1(A) = \lim_{n\to\infty}[m(A^n)]^{1/n}$. If A is invertible, then $r_1(A) = [r(A^{-1})]^{-1}$. Also let $r(A)$ denote the spectral radius of A.

If T is an injective unilateral weighted shift with weight sequence $\{w_n\}$, then we define $r_2(T)$ by $r_2(T) = \liminf_{n\to\infty}[w_0 w_1 \cdots w_{n-1}]^{1/n}$. Thus $r_1(T) \leq r_2(T) \leq r(T)$.

In order to characterize the unilateral weighted shifts in $A_1(G)$, we first let $G = \{z: |z| < r_1(T)\}$; then

(a) $G \subseteq \sigma_p(T^*)$ [**5**, Theorem 8(ii)],

(b) $G \cap \sigma_{ap}(T) = \square$ [**5**, Theorem 6],

(c) $\sigma_p(T) = \square$ [**5**, Theorem 8(i)],

(d) If $f \in \bigcap_{\lambda \in G} \operatorname{ran}(T - \lambda)$, then $f \equiv 0$ [**5**, Theorem (ii)], and

(e) $\dim \ker(T^* - \bar{\lambda}) = 1$, $\lambda \in G$ [**5**, Theorem 8]. However, $\sigma(T)$ might not be equal to $\bar{G}$ unless $r_1(T) = r(T)$, in which case $r_1(T) = r_2(T) = r(T)$.

Next let T be any unilateral weighted shift in $A_1(G)$. Then $\bar{G} = \sigma(T) = \{z: |z| \leq r(T)\}$, so $G \subseteq \{z: |z| < r(T)\}$. Since $G \cap \sigma_{ap}(T) = \square$, we have $G \subseteq \{z: |z| < r_1(T)\}$. It follows that $\bar{G} \subseteq \{z: |z| \leq r_1(T)\}$. But $\bar{G} = \{z: |z| \leq r(T)\}$, and from this one obtains that $r_1(T) = r(T)$. Now $G_T = \operatorname{int}[\sigma(T) \sim \sigma_{ap}(T)] = \{z: |z| < r(T)\}$.

Note. If T is a unilateral weighted shift in $A_1(G)$, then the set G_T is actually independent of T. To see this let T and T' be unilateral weighted shifts in $A_1(G)$. Then $\sigma(T) = \sigma(T') = \bar{G}$, so $r(T) = r(T')$. We have also shown that $r(T) = r_1(T)$ and $r(T') = r_1(T')$. Therefore $r(T) = r(T') = r_1(T') = r_1(T)$. Hence $G_T = G_{T'}$.

We have therefore proved the following.

(2.1) THEOREM. *Let T be a unilateral weighted shift. Then T is in $A_1(G)$ if and only if $\bar{G} = \{z: |z| \leq r(T)\}$ and $r(T) = r_1(T)$.*

If T is an injective bilateral weighted shift with weight sequence $\{w_n\}$ then define $r_2^+(T) = \liminf_{n\to\infty}[w_0 \cdots w_{n-1}]^{1/n}$, $r_3^+(T) = \limsup_{n\to\infty}[w_0 \cdots w_{n-1}]^{1/n}$, $r_2^-(T) = \liminf_{n\to\infty}[w_{-1} \cdots w_{-n}]^{1/n}$, $r_3^-(T) = \limsup_{n\to\infty}[w_{-1} \cdots w_{-n}]^{1/n}$. Then $r_1^- \leq r_2^- \leq r_3^- \leq r^-$ and $r_1^+ \leq r_2^+ \leq r_3^+ \leq r^+$.

In order to characterize the bilateral weighted shifts in $A_1(G)$, we first let $G = \{z: r_1(T) < |z| < r(T)\}$ and assume $r_1 = r_1^- = r^-$, $r_1^+ = r^+ = r$ $(r_1 < r)$. It follows that

(a) $G \subseteq \sigma_p(T^*)$ [**5**, Theorem 9(iii)],

(b) $G \cap \sigma_{ap}(T) = \square$ [**5**, Theorem 7],

(c) $\sigma_p(T) = \square$ [**5**, Theorem 9(ii)],

(d) if $f \in \bigcap_{\lambda \in G} \operatorname{ran}(T - \lambda)$, then $f \equiv 0$ [**5**, Theorem 10′(ii)],

(e) $\dim \ker(T^* - \bar{\lambda}) = 1$, $\lambda \in G$ [**5**, Theorem 9(i)],

(f) $\sigma(T) \subseteq \bar{G}$ [**5**, Theorem 5(a)].

Next let T be any bilateral weighted shift in $A_1(G)$. If T is not invertible, then $\sigma(T) = \{z: |z| \leq r(T)\}$. Since $G \cap \sigma_{ap}(T) = \square$, $r^- < r_1^+$ and $\sigma_{ap}(T) = \{r_1^- \leq |z| \leq r^-\} \cup \{r_1^+ \leq |z| \leq r^+\}$. Now $r_1^- \leq r^- < r_1^+ \leq r^+$, so $G \subset \{|z| < r_1^-\} \cup \{r^- < |z| < r_1^+\}$. From this we get $\bar{G} \subseteq \{|z| \leq r_1^-\} \cup \{r^- \leq |z| \leq r_1^+\}$. But $\bar{G} \subseteq \{|z| \leq r(T)\}$, so $r_1^- = r^- = r_1^+ = r^+$. Equivalently $r_1 = r$. Thus $r_1^- = r_2^- = r_3^- = r = r_1^+ = r_2^+ = r_3^+$. It follows that $\sigma_p(T^*) = \{|z| = r\}$. Therefore (a) does not hold. We have shown that if T is a bilateral weighted shift in $A_1(G)$ it is necessary that T be invertible. In this case $\sigma(T) = \{r_1 \leq |z| \leq r\}$, $\sigma_{ap}(T) = \{r_1^- \leq |z| \leq r^-\} \cup \{r_1^+ \leq |z| \leq r^+\}$, and $r_1^- \leq r^- \leq r_1^+ \leq r^+$. Since $G \cap \sigma_{ap}(T) = \square$, it is easy to see that the only possibility is $r_1^- = r^-$ and $r_1^+ = r^+$. So $\sigma_{ap}(T) = \{|z| = r\} \cup \{|z| = r_1\}$. We also have $G_T = \operatorname{int}[\sigma(T) \sim \sigma_{ap}(T)] = \{z: r_1 < |z| < r\}$.

Note. It is easy to see that G_T does not depend on T. We have thus proved the following.

(2.2) THEOREM. *If T is a bilateral weighted shift, then T is in $A_1(G)$ if and only if $\bar{G} = \{z: r_1(T) \leq |z| \leq r(T)\}$, $r_1^- = r^-$ and $r_1^+ = r^+$.*

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