ESSENTIAL SPECTRA OF OPERATORS IN THE CLASS $\mathfrak{B}_n(\Omega)$

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Abstract. For a connected open subset $\Omega$ of the plane and $n$ a positive integer, let $\mathfrak{B}_n(\Omega)$ be the space introduced by Cowen and Douglas in their paper Complex Geometry and Operator Theory. Our paper deals with characterizing the essential spectrum of an operator $T$ in $\mathfrak{B}_n(\Omega)$ for which $\sigma(T) = \overline{\Omega}$ and the point spectrum of $T^*$ is empty. This class of operators forms an important part of $\mathfrak{B}_n(\Omega)$ denoted by $\mathfrak{B}_n^0(\Omega)$. We use this characterization to give another proof of the result of Axler, Conway and McDonald on determining the essential spectrum of the Bergman operator.

Let $A_n(G) = \{S: T = S^* is in \mathfrak{B}_n^0(G^*)\}$. We also characterize the weighted shifts in $A_n(G)$.

This paper deals with characterizing the essential spectrum of an operator $T$ in $\mathfrak{B}_n^0(\Omega)$ for which $\sigma(T) = \overline{\Omega}$ and the point spectrum of $T^*$ is empty. This class of operators is denoted by $\mathfrak{B}_n^0(\Omega)$. We use this characterization to give another proof of the result of Axler, Conway and McDonald on determining the essential spectrum of the Bergman operator.

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Notation and terminology. The following notation will be retained throughout the rest of this paper:

$\emptyset = \text{empty set}, \quad \mathbb{C} = \text{complex plane}$.

For a subset $E$ of the plane, $\partial E$ is the boundary of $E$, $\text{int } E$ (or $E^\circ$) is the interior of $E$, and $\overline{E}$ is the closure of $E$.

$L(\mathcal{H})$ will denote the collection of bounded linear operators on the complex Hilbert space $\mathcal{H}$.

For $T$ in $L(\mathcal{H})$, the spectrum of $T$, point spectrum of $T$, and approximate point spectrum of $T$ will be denoted by $\sigma(T)$, $\sigma_p(T)$, and $\sigma_{app}(T)$ respectively. See [4] for the appropriate definitions.

An operator $T$ in $L(\mathcal{H})$ is called a Fredholm operator if $\text{ran } T$ is closed and both $\ker T$ and $\ker T^*$ are finite dimensional. The essential spectrum of $T$, denoted by $\sigma_e(T)$, is the set of all $\lambda$ in $\mathbb{C}$ such that $T - \lambda$ is not Fredholm.

The closed linear span of a family of subspaces $\{\mathfrak{H}_a\}$ is $V_a\mathfrak{H}_a$.
If $B$ is a subset of the complex plane, then $B^* = \{ z : \bar{z} \in B \}$. Finally $B(z, \delta) = \{ \lambda \in \mathbb{C} : |z - \lambda| < \delta \}$.

1. Essential spectrum. For a connected open subset $\Omega$ of the plane and $n$ a positive integer, let $\mathcal{B}_n(\Omega)$ denote the operators $T$ defined on the Hilbert space $\mathcal{H}$ which satisfy

- (a) $\Omega \subseteq \sigma(T) = \{ \omega \in \mathbb{C} : T - \omega$ is not invertible$\}$,
- (b) $\text{ran}(T - \omega) = \mathcal{H}$ for $\omega$ in $\Omega$,
- (c) $\forall \omega \in \Omega \ker(T - \omega) = \mathcal{H}$, and
- (d) $\dim \ker(T - \omega) = n$ for $\omega$ in $\Omega$.

The space $\mathcal{B}_n(\Omega)$ has been introduced and investigated by Cowen and Douglas [3].

In this section we want to determine the essential spectrum of operators in $\mathcal{B}_n(\Omega)$, where $\mathcal{B}_n(\Omega) = \{ T \in \mathcal{B}_n(\Omega) : \sigma(T) = \overline{\Omega}$ and $\sigma_p(T^*) = \emptyset \}$, and, as a consequence, we deduce the result of Axler, Conway and McDonald [1] on characterizing the essential spectrum of the Bergman operator.

(1.1) Theorem. Let $\Omega$ be an open connected subset of the plane, let $n$ be a positive integer, and let $T \in \mathcal{B}(\mathcal{H})$ such that

- (i) $\sigma(T) = \overline{\Omega}$,
- (ii) $\text{ran}(T - \omega)$ is closed, $\omega \in \Omega$,
- (iii) $\dim \ker(T - \omega) = n$, $\omega \in \Omega$,
- (iv) $\sigma_p(T^*) = \emptyset$.

Let $\Omega_0 = \{ \omega_0 \in \overline{\Omega} : \text{ran}(T - \omega_0)$ is closed and there exist holomorphic $\mathcal{H}$-valued functions $\{ e_i(\omega) \}_{i=1}^n$ defined on some neighborhood $B$ of $\omega_0$ such that for each $\omega$ in $B$ $\{ e_i(\omega) \}_{i=1}^n$ forms a basis for $\ker(T - \omega)$}. Then $\sigma_e(T) = \partial \Omega_0$.

Proof. Let $\Omega' = \{ \omega_0 \in \overline{\Omega} : \text{ran}(T - \omega_0)$ is closed and $\dim \ker(T - \omega) = n$ for $\omega$ in some neighborhood $B$ of $\omega_0 \}$. Note that $\Omega'$ is open and $\Omega \subseteq \Omega' \subseteq \overline{\Omega}$. Hence the closure of $\Omega'$ is $\overline{\Omega}$. Also for $\omega \in \Omega'$, $T - \omega$ is Fredholm and $\text{ind}(T - \omega) = n$. We show that $\sigma_e(T) = \partial \Omega'$ and $\Omega' = \Omega_0$.

If $\omega \notin \partial \Omega'$, either $\omega$ is not in the closure of $\Omega'$ (hence $\omega \notin \overline{\Omega}$), in which case $T - \omega$ is invertible and hence Fredholm, or $\omega$ is in the closure of $\Omega'$, in which case $\omega \in \partial \Omega'$ and $T - \omega$ is Fredholm. So $\sigma_e(T) \subseteq \partial \Omega'$. If $\omega_0 \notin \sigma_e(T)$, then $T - \omega_0$ is Fredholm; thus $T - \omega$ is Fredholm for $\omega$ in some neighborhood $B$ of $\omega_0$ and $\text{ind}(T - \omega)$ is constant on this neighborhood. If $B \cap \Omega' = \emptyset$, then $\omega_0 \notin \partial \Omega'$. If $B \cap \Omega' \neq \emptyset$, then $\text{ind}(T - \omega) = n$, $\omega \in B$. Because $\sigma_p(T^*) = \emptyset$, $\dim \ker(T - \omega) = n$, $\omega \in B$. Thus $B \subseteq \Omega'$, so $\omega_0 \notin \partial \Omega'$. It follows that $\sigma_e(T) = \partial \Omega'$.

The inclusion $\Omega' \subseteq \Omega_0$ follows from Šubin's Theorem ([3], Cowen and Douglas). Also, it is easy to see that $\Omega_0 \subseteq \Omega'$. Hence $\Omega_0 = \Omega'$ and $\sigma_e(T) = \partial \Omega_0$. Q.E.D.

We now have proved the following.

(1.2) Corollary. Let $T \in \mathcal{B}_n(\Omega)$ and let $\Omega_0$ be as in Theorem 1.1. Then $\sigma_e(T) = \partial \Omega_0$.

For the rest of our discussion let $G$ be a bounded, open, connected, nonempty subset of the complex plane $\mathbb{C}$ and let $L^2(G)$ denote the $L^2$-space of (Lebesgue) area measure restricted to $G$. Denote by $L^2_a(G)$ the subspace of analytic functions
belonging to $L^2(G)$. The Bergman space $L^2_a(G)$ is actually a closed subspace of $L^2(G)$ and thus is a Hilbert space. Let $T_\zeta$ be the Bergman operator on $L^2_a(G)$ defined by $T_\zeta f = \overline{f}$.

(1.3) Definition. A point $\lambda \in \partial G$ is said to be removable with respect to $L^2_a(G)$ if there exists an open neighborhood $V$ of $\lambda$ such that every function in $L^2_a(G)$ can be extended to an analytic function on $G \cup V$. The set of all points on $\partial G$ which are removable with respect to $L^2_a(G)$ is denoted by $\partial_{\sim}G$. The Bergman essential boundary of $G$, denoted by $\partial_{\sim}E_G$, is the set of all points of $\partial G$ which are not removable with respect to $L^2_a(G)$; so $\partial_{\sim}E_G = \partial G \sim \partial_{\sim}G$.

It is shown in [1] that $G \cup \partial_{\sim}G$ is an open subset of $\mathbb{C}$ and that $\partial_{\sim}E_G$ has zero area. Therefore it makes sense to consider $L^2_a(G \cup \partial_{\sim}G)$ and it is easy to see that $L^2_a(G) = L^2_a(G \cup \partial_{\sim}G)$. For $\lambda \in G \cup \partial_{\sim}G$, the linear functional on $L^2_a(G \cup \partial_{\sim}G)$ which takes $h$ to $h(\lambda)$ is bounded (see [2, p. 5]).

(1.4) Theorem. Let $G_0 = \{ \lambda_0 \in \overline{G} : \text{ran}(T_{\zeta} - \lambda_0) \text{ is closed and there exists a conjugate holomorphic } L^2_a(G)\text{-valued function } d(\lambda) \text{ defined on some neighborhood } V \text{ of } \lambda_0 \text{ such that } \ker(T_{\zeta}^* - \lambda) = [d(\lambda)] \text{ for } \lambda \in V \}$. Then $G_0 = G \cup \partial_{\sim}G$. Equivalently $\sigma_e(T_\zeta) = \partial_{\sim}E_G$.

Proof. First suppose that $\lambda \in G \cup \partial_{\sim}G$. Then it is easy to verify that the range of $T_{\zeta} - \lambda$ is equal to the kernel of the linear functional on $L^2_a(G) = L^2_a(G \cup \partial_{\sim}G)$ which sends $h$ to $h(\lambda)$. In particular, the range of $T_{\zeta} - \lambda$ is a closed subspace of $L^2_a(G)$ of codimension 1. Since $\ker(T_{\zeta} - \lambda) = (0)$, we conclude that $T_{\zeta} - \lambda$ is Fredholm (with index -1) and so $\lambda \in \sigma_e(T_\zeta)$. Thus $G \cup \partial_{\sim}G \subseteq \overline{G} \sim \sigma_e(T_\zeta)$.

Next choose $\lambda_0 \in G_0$. Let $h(\lambda) = (1, d(\lambda))$. If $h(\lambda) \equiv 0$ for $\lambda$ in a neighborhood of $\lambda_0$, then $h$ is holomorphic in a neighborhood of $\lambda_0$ and does not vanish identically there. Hence there exists $r > 0$ such that $B(\lambda_0, r) \subseteq G_0$ and for $\lambda$ in $B(\lambda_0, r)$, $h$ is holomorphic and $h(\lambda) \neq 0$ for $\lambda$ in $B_0 = B(\lambda_0, r) \sim \{\lambda_0\}$.

Now for $\lambda$ in $B_0$ define $k(\lambda) = h(\lambda)^{-1}d(\lambda)$. Obviously $k(\lambda), \lambda \in B_0$ has the same properties as $d(\lambda)$. Furthermore it is normalized such that $(1, k(\lambda)) = 1, \lambda \in B_0$. Let $\{e(\lambda)\}$ be the Bergman kernel (reproducing kernel) for $L^2_a(G)$. Then for $\lambda \in G \cap B_0$, $\ker(T_{\zeta}^* - \lambda) = [e(\lambda)] = [k(\lambda)]$. Hence $e(\lambda) = c(\lambda)k(\lambda)$, where $c(\lambda) \in \mathbb{C}$. By normalization, $1 = (1, e(\lambda)) = c(\lambda)(1, k(\lambda)) = c(\lambda)$ and therefore $e(\lambda) = k(\lambda)$. If $f \in L^2_a(G)$, then $f(\lambda) = (f, e(\lambda))$, $\lambda \in G$. Now define $\tilde{f}(\lambda) = (f, k(\lambda))$, $\lambda \in B_0$. Clearly $\tilde{f}(\lambda) = f(\lambda), \lambda \in B_0 \cap G$. Hence $B_0 \subseteq G \cup \partial_{\sim}G$. If $f \in L^2_a(G) = L^2_a(G \cup \partial_{\sim}G)$, then $f|_{B_0}$ is in $L^2_a(B_0)$ because $\partial_{\sim}G$ has zero area [1]. However, a function in $L^2_a(B_0)$ extends to $B[1]$. Thus each function in $L^2_a(G)$ extends to be analytic on $B$. This shows that $\lambda_0 \not\in \partial_{\sim}G$. Hence $G_0 \subseteq G \cup \partial_{\sim}G$.

We will show that $G_0 = \overline{G} \sim \sigma_e(T_\zeta)$. To see this let $\Omega = G^* \cap T = T_{\zeta}^*$. Then $T$ satisfies the hypothesis of Theorem 1.1 with $n = 1$. Therefore $\sigma_e(T) = \partial\Omega_0$. It is easy
to see that $\Omega_0 = G_0^*$. Hence $\sigma(T) = \sigma(T)^* = \sigma(T) = (\partial\Omega_0)^* = \partial\Omega_0^* = \partial G_0$. Since $G_0$ is open, $G \sim \sigma(T) = \overline{G} \sim \partial G_0 = G_0$.

We have already shown that $G_0 \subseteq G \cup \partial\mathcal{G} \subseteq \overline{G} \sim \sigma(T)$; therefore $G_0 = G \cup \partial\mathcal{G} = \overline{G} \sim \sigma(T)$. We conclude that $\sigma(T) = \partial\mathcal{G}$. Q.E.D.

2. Weighted shifts in $A_1(G)$. For a connected open subset $G$ of the plane and $n$ a positive integer, let $A_n(G)$ denote the operators $S$ in $\mathcal{E}(\mathcal{H})$ which satisfy

(a) $G \subseteq \sigma_p(S^*)$, 
(b) $G \cap \sigma_{ap}(S) = \emptyset$, 
(c) $\sigma_p(S) = \emptyset$, 
(d) $\bigcap_{\lambda \in G} \text{ran}(S - \lambda) = \{0\}$, 
(e) $\dim \ker(S^* - \lambda) = n, \lambda \in G$, and 
(f) $\sigma(S) = \overline{G}$.

The space $A_n(G)$ is closely connected with the space $\mathcal{B}_n(G)$. In particular it is easy to see that $A_n(G) = \{S: T = S^* \in \mathcal{B}_n(G^*)\}$.

A unilateral (bilateral) weighted shift on a Hilbert space $\mathcal{H}$ is a linear map $T$ from $\mathcal{H}$ to itself given by $T e_i = w_i e_{i+1}$, where $(e_i) = (e_i)_{i=0}^\infty ((e_i)_{i=-\infty}^\infty)$ is an orthonormal basis for $\mathcal{H}$ and $\sup_i |w_i| < \infty$.

In his survey article, A. L. Shields [5] has investigated the properties of weighted shifts which we need in order to characterize the weighted shifts in $A_1(G)$. For a more detailed treatment of the subject and pertinent terminology see [5]. We now make a few observations.

Let $S$ be in $A_n(G)$ and note that $\sigma(S) \sim \sigma_{ap}(S) = (\partial\sigma(S) \sim \sigma_{ap}(S)) \cup \{(\sigma(S))^0 \sim \sigma_{ap}(S))$. Because $\partial\sigma(S) \subseteq \sigma_{ap}(S)$ and $\sigma_{ap}(S)$ is closed (see Halmos [4, problem 62]), we conclude that $\sigma(S) \sim \sigma_{ap}(S)$ is an open set. Put $G_S = \sigma(S) \sim \sigma_{ap}(S)$; then $G \subseteq G_S \subseteq \sigma(S)$, so $\sigma(S) = \overline{G} = \overline{G_S}$. If $\lambda \in G_S$, then $\lambda \not\in \sigma_{ap}(S)$, so $S - \lambda$ is bounded below. Therefore $S - \lambda$ has closed range. Also $\ker(S - \lambda) = \{0\}$, from which we conclude that $S - \lambda$ is semi-Fredholm. Now choose a sequence $\{\lambda_k\}$ in $G$ such that $\lambda_k \to \lambda$; then $\text{ind}(S - \lambda_k) \to \text{ind}(S - \lambda)$. Now for $S$ in $A_n(G)$ and $\lambda \in G$, $S - \lambda$ is a Fredholm operator and $\text{ind}(S - \lambda) = -n$. Therefore $\text{ind}(S - \lambda) = -n$ and we conclude that $S \in A_n(G_S)$. Clearly $G_S$ is the largest open subset $G$ of the plane such that $S \in A_n(G)$.

If $A$ is any operator let $m(A) = \inf\{\|Af\|: \|f\| = 1\}$. Then the following is true (see Shields [5]):

$$\sup_n \left[ m(A^n) \right]^{1/n} = \lim_{n \to \infty} \left[ m(A^n) \right]^{1/n}. $$

Let $r_1(A) = \lim_{n \to \infty} [m(A^n)]^{1/n}$. If $A$ is invertible, then $r_1(A) = [r(A^{-1})]^{-1}$. Also let $r(A)$ denote the spectral radius of $A$.

If $T$ is an injective unilateral weighted shift with weight sequence $\{w_n\}$, then we define $r_2(T)$ by $r_2(T) = \lim \inf_{n \to \infty} [w_0 w_1 \cdots w_{n-1}]^{1/n}$. Thus $r_1(T) \leq r_2(T) \leq r(T)$.

In order to characterize the unilateral weighted shifts in $A_1(G)$, we first let $G = \{z: \|z\| < r_1(T)\}$; then

(a) $G \subseteq \sigma_p(T^*)$ [5, Theorem 8(ii)],

(b) $G \cap \sigma_{ap}(T) = \emptyset$ [5, Theorem 6].
(c) \( \sigma_p(T) = \emptyset \) [5, Theorem 8(ii)].

(d) If \( f \in \bigcap_{\lambda \in G} \text{ran}(T - \lambda) \), then \( f \equiv 0 \) [5, Theorem (ii)], and

(e) \( \dim \ker(T^* - \lambda) = 1, \lambda \in G \) [5, Theorem 8]. However, \( \sigma(T) \) might not be equal to \( \tilde{G} \) unless \( r_1(T) = r(T) \), in which case \( r_1(T) = r_2(T) = r(T) \).

Next let \( T \) be any unilateral weighted shift in \( A_1(G) \). Then \( \tilde{G} = \sigma(T) = \{ z : |z| \leq r(T) \} \), so \( G \subseteq \{ z : |z| < r(T) \} \). Since \( G \cap \sigma_{ap}(T) = \emptyset \), we have \( G \subseteq \{ z : |z| < r(T) \} \). It follows that \( \tilde{G} \subseteq \{ z : |z| < r_1(T) \} \). But \( \tilde{G} = \{ z : |z| < r(T) \} \), and from this one obtains that \( r_1(T) = r(T) \). Now \( G_T = \text{int}[\sigma(T) - \sigma_{ap}(T)] = \{ z : |z| < r(T) \} \).

**Note.** If \( T \) is a unilateral weighted shift in \( A_1(G) \), then the set \( G_T \) is actually independent of \( T \). To see this let \( T \) and \( T' \) be unilateral weighted shifts in \( A_1(G) \). Then \( \sigma(T) = \sigma(T') = \tilde{G} \), so \( r(T) = r(T') \). We have also shown that \( r(T) = r_1(T) \) and \( r(T') = r_1(T') \). Therefore \( r(T) = r(T') \). Hence \( G_T = G_T \).

We have therefore proved the following.

(2.1) **Theorem.** Let \( T \) be a unilateral weighted shift. Then \( T \) is in \( A_1(G) \) if and only if \( G = \{ z : |z| < r(T) \} \) and \( r(T) = r_1(T) \).

If \( T \) is an injective bilateral weighted shift with weight sequence \( \{ w_n \} \) then define
\[
\begin{align*}
r_2^+(T) &= \liminf_{n \to \infty} [w_0 \cdots w_{n-1}]^{1/n}, \\
r_3^+(T) &= \limsup_{n \to \infty} [w_0 \cdots w_{n-1}]^{1/n}. \\
r_2^-(T) &= \liminf_{n \to \infty} [w_1 \cdots w_n]^{1/n}, \\
r_3^-(T) &= \limsup_{n \to \infty} [w_1 \cdots w_n]^{1/n}. \\
r_2^-(T) &= \liminf_{n \to \infty} [w_0 \cdots w_{n-1}]^{1/n}.
\end{align*}
\]
Then \( r_2^+ \leq r_3^- \leq r_1^- \leq r_2^- \leq r_3^+ \). In order to characterize the bilateral weighted shifts in \( A_1(G) \), we first let \( G = \{ z : r_1(T) < |z| < r(T) \} \) and assume \( r_1 = r_1^- = r^- \). It follows that
(a) \( G \subseteq \sigma_p(T^*) \) [5, Theorem 9(iii)],
(b) \( G \cap \sigma_{ap}(T) = \emptyset \) [5, Theorem 7],
(c) \( \sigma_p(T) = \emptyset \) [5, Theorem 9(ii)],
(d) if \( f \in \bigcap_{\lambda \in G} \text{ran}(T - \lambda) \), then \( f \equiv 0 \) [5, Theorem 10(ii)],
(e) \( \dim \ker(T^* - \lambda) = 1, \lambda \in G \) [5, Theorem 9(i)],
(f) \( \sigma(T) \subseteq \tilde{G} \) [5, Theorem 5(a)].

Next let \( T \) be any bilateral weighted shift in \( A_1(G) \). If \( T \) is not invertible, then \( \sigma(T) = \{ z : |z| \leq r(T) \} \). Since \( G \cap \sigma_{ap}(T) = \emptyset \), \( r^- < r_1^+ \). Now \( r_1^- < r^- < r_1^+ < r^+ \), so \( G \subseteq \{|z| < r_1^- \} \cup \{|z| < r^- \} \). From this we get \( \tilde{G} \subseteq \{|z| < r_1^- \} \cup \{|z| < r^- \} \). But \( \tilde{G} \subseteq \{|z| < r(T) \} \), so \( r_1^- = r^- = r_1^+ = r^+ \). Equivalently \( r_1 = r \). Thus \( r_1^- = r_1^+ = r_3^- = r_3^+ = r_2^+ = r_2^- = r_2^+ = r_2^- \). It follows that \( \sigma_{ap}(T) = \{|z| = r \} \). Therefore (a) does not hold. We have shown that if \( T \) is a bilateral weighted shift in \( A_1(G) \) it is necessary that \( T \) be invertible. In this case \( \sigma(T) = \{|r_1^+ < |z| < r \} \), \( \sigma_{ap}(T) = \{|r_1^- < |z| < r^- \} \cup \{|r_1^+ < |z| < r^- \} \cup \{|r_1^- < |z| < r^+ \} \), and \( r_1^- < r^- < r_1^+ < r^+ \). Since \( G \cap \sigma_{ap}(T) = \emptyset \), it is easy to see that the only possibility is \( r_1^- = r^- \) and \( r_1^+ = r^+ \). So \( \sigma_{ap}(T) = \{|z| = r \} \cup \{|z| = r_1 \} \). We also have \( G_T = \text{int}[\sigma(T) - \sigma_{ap}(T)] = \{ z : r_1^- < |z| < r \} \).

**Note.** It is easy to see that \( G_T \) does not depend on \( T \). We have thus proved the following.

(2.2) **Theorem.** If \( T \) is a bilateral weighted shift, then \( T \) is in \( A_1(G) \) if and only if \( G = \{ z : r_1(T) < |z| < r(T) \} \), \( r_1^- = r^- \) and \( r_1^+ = r^+ \).
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