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DIVERGENT JACOBI POLYNOMIAL SERIES

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ABSTRACT. Fix real numbers $\alpha \geqslant \beta \geqslant -\frac{1}{2}$, with $\alpha > -\frac{1}{2}$, and equip $[-1, 1]$ with the measure $d\mu(x) = (1-x)^\alpha(1+x)^\beta\,dx$. For $p = 4(\alpha+1)/(2\alpha+3)$ there exists $f \in L^p(\mu)$ such that $f(x) = 0$ a.e. on $[-1, 0]$ and the appropriate Jacobi polynomial series for f diverges a.e. on $[-1, 1]$. This implies failure of localization for spherical harmonic expansions of elements of $L^{2d/(d+1)}(X)$, where X is a sphere or projective space of dimension $d > 1$.

In 1972 H. Pollard [**11**] raised the following question: Is there an $f \in L^{4/3}([-1, 1])$ whose Legendre polynomial series diverges almost everywhere? We show that the answer is yes and that this is a special case of a divergence result for series of Jacobi polynomials. This general result, which we state below, is a consequence of some estimates of J. Newman and W. Rudin [**10**], the uniform boundedness principle, and the Cantor-Lebesgue theorem. As a corollary, we present examples of the failure of localization for series of spherical harmonics on spheres and projective spaces.

Fix real numbers $\alpha \geqslant \beta \geqslant -\frac{1}{2}$, with $\alpha > -\frac{1}{2}$, and let μ denote the measure on $[-1, 1]$ defined by

$$d\mu(x) = (1-x)^\alpha(1+x)^\beta\,dx.$$

Let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial of degree n associated to (α, β), as described in G. Szegö's book [**13**, Chapter 4]. Furthermore, fix

$$p = 4(\alpha+1)/(2\alpha+3) \quad \text{and} \quad p' = 4(\alpha+1)/(2\alpha+1),$$

and note that $1 < p < 2$ and $(1/p) + (1/p') = 1$.

If we set $h_n^{(\alpha,\beta)} = \int_{-1}^1 |P_n^{(\alpha,\beta)}|^2\,d\mu$, for $n \geqslant 0$, and if $f \in L^1(\mu)$, then f has Jacobi polynomial series

$$\sum_{n=0}^{\infty} c_n\left(h_n^{(\alpha,\beta)}\right)^{-1} P_n^{(\alpha,\beta)}(x),$$

where the coefficients c_n are given by

$$c_n = \int_{-1}^{1} f.P_n^{(\alpha,\beta)}\,d\mu.$$

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We will prove the following result:

1. THEOREM. *For* $a > -\frac{1}{2}$, $\alpha \geqslant \beta \geqslant -\frac{1}{2}$ *and* $p = 4(\alpha+1)/(2\alpha+3)$ *there is an element* $f \in L^p([-1,1],\mu)$, *supported in* $[0,1]$, *whose Jacobi polynomial series* $\sum_{n=0}^{\infty} c_n (h_n^{(\alpha,\beta)})^{-1} P_n^{(\alpha,\beta)}$ *diverges almost everywhere on* $[-1,1]$.

2. REMARKS. If $\alpha = \beta = 0$, then $d\mu(x) = dx$, $p = 4/3$, and $P_n^{(0,0)}(x) = P_n(x)$. Hence the theorem answers Pollard's question. See also [**4** and **7**].

The theorem also shows that there is no localization principle for $L^p(\mu)$, unlike the case of trigonometric Fourier series of Lebesgue-integrable functions on $[-1,1]$.

We first recall the result of [**10**].

3. PROPOSITION. *For* α, β *and* p' *as above and* $n > 1$,

$$\int_{-1}^{1} | P_n^{(\alpha,\beta)} |^{p'} d\mu(x) \geqslant C(\log n) \cdot n^{-p'/2},$$

where C is a positive constant depending on α *and* β.

The uniform boundedness principle then leads to the following result:

4. LEMMA. *For* α, β *and p as above and for any sequences* $\{\varepsilon_n\}_{n=0}^{\infty}$, *with* $\varepsilon_n > 0$ *and* $\lim_{n\to\infty} \varepsilon_n = 0$, *there exists* $f \in L^p(\mu)$ *with*

$$\limsup_{n\to\infty} | c_n \cdot n^{1/2} \varepsilon_n^{-1} (\log n)^{-1/p'} | = \infty.$$

Furthermore, we can suppose that $f = 0$ *on* $[-1,0]$.

PROOF. From Proposition 3, the fact that $|P_n^{(\alpha,\beta)}(-x)| = |P_n^{(\beta,\alpha)}(x)|$ and the hypothesis $\alpha \geqslant \beta$, we see that

$$\left(\int_0^1 | P_n^{(\alpha,\beta)} |^{p'} d\mu \right)^{1/p'} \geqslant C \cdot (\log n)^{1/p'} \cdot n^{-1/2}.$$

The left-hand side of this inequality is the norm of the linear functional $f \to c_n = \int_0^1 f.P_n^{(\alpha,\beta)}\, d\mu$, defined on $L^p([0,1],\mu)$. Q.E.D.

We next need to know the asymptotic properties of $P_n^{(\alpha,\beta)}(x)$. See [**13**, Theorem 8.21.8].

5. PROPOSITION. *For* $\alpha \geqslant \beta \geqslant -\frac{1}{2}$ *and* $\varepsilon > 0$ *the following estimate holds uniformly for* $\varepsilon \leqslant \theta \leqslant \pi - \varepsilon$ *and all* $n \geqslant 1$:

$$P_n^{(\alpha,\beta)}(\cos\theta) = n^{-1/2} k(\theta) \cos(N\theta + \gamma) + \mathbf{O}(n^{-3/2}),$$

where $k(\theta) = \pi^{-1/2}(\sin(\theta/2))^{-\alpha-1/2} \cdot (\cos(\theta/2))^{-\beta-1/2}$, $N = n + \frac{1}{2}(\alpha+\beta+1)$, *and* $\gamma = -(\alpha+\frac{1}{2})\pi/2$.

Observe that $h_n^{(\alpha,\beta)} \sim C_{\alpha,\beta} n^{-1}$ [**13**, (4.3.3)]. Combining this with Egoroff's theorem shows that if

$$\sum_{n=0}^{\infty} c_n \left(h_n^{(\alpha,\beta)}\right)^{-1} P_n^{(\alpha,\beta)}(x)$$

converges on a set of positive measure in $[-1, 1]$ then there is a subset, say E, of positive measure such that

$$c_n \cdot n^{1/2}\big(\cos(N\theta + \gamma) + \mathbf{O}(n^{-1})\big) \to 0, \quad \text{as } n \to \infty,$$

uniformly for $\cos\theta \in E$.

6. LEMMA. *If $\{c_n\}$ is a sequence such that*

$$c_n \cdot n^{1/2}\big(\cos(N\theta + \gamma) + \mathbf{O}(n^{-1})\big) \to 0, \quad \text{as } n \to \infty,$$

for θ in a set of positive measure, then

$$c_n \cdot n^{1/2} \to 0 \quad \text{as } n \to \infty.$$

PROOF. Argue as in the proof of the Cantor-Lebesgue theorem, as described in [**14**, p. 316].

We can now complete the proof of the theorem by observing that if $\varepsilon_n = (\log n)^{-1/(2p')}$ in Lemma 4 then there exists $f \in L^p([0, 1], \mu)$ with

$$\limsup_{n \to \infty} |c_n| \cdot n^{1/2}(\log n)^{-1/(2p')} = \infty.$$

Lemma 6 then shows that this sequence $\{c_n\}_n$ must have $\sum_{n=0}^{\infty} c_n(h_n^{(\alpha,\beta)})^{-1}P_n^{(\alpha,\beta)}(x)$ divergent almost everywhere. A similar idea is employed in [**12**] to demonstrate divergence of central Fourier series on compact Lie groups.

7. REMARKS. For other results concerning convergence of Jacobi polynomial series see [**3**, **5** and **9**]. When $\alpha = \beta = \frac{1}{2}$, Theorem 1 is a result concerning failure of localization for Fourier series of central elements of $L^{3/2}(SU(2))$.

More generally, let X be a compact two-point homogeneous space [**6**] of dimension $d > 1$. Hence, X is one of the spaces S^m, $P^m(\mathbf{R})$, $P^m(\mathbf{C})$, $P^m(\mathbf{H})$, or P^2 (Cayley). Equip X with the canonical measure ω, Laplace-Beltrami operator Δ, and metric ρ associated with its Riemannian structure. Suppose that X has diameter l. Let $0 = \lambda_0 < \lambda_1 < \cdots$ denote the distinct eigenvalues of $-\Delta$, with corresponding eigenspaces $\mathcal{H}_n$. Any integrable function f on X has an eigenfunction expansion

$$\sum_{n=0}^{\infty} Y_n(f\colon x),$$

with $Y_n(f) \in \mathcal{H}_n$, $\forall n \geq 0$. When $X = S^d$, this is the usual spherical harmonic expansion of a function on the sphere.

A function f on X is said to be *zonal* about a point x_0 if $f(y)$ depends only on $\rho(y, x_0)$, for all $y \in X$. It is well known that zonal elements of $\mathcal{H}_n$ are described in terms of Jacobi polynomials of degree n. See [**1** and **2**] and the references listed there.

The author has shown [**8**] that if $f \in L^2(X, \omega)$ and if $f = 0$ a.e. on an open set $U \subset X$ then $\sum_{n=0}^{\infty} Y_n(f\colon x)$ converges to zero almost everywhere on U. Using properties of Jacobi polynomials, [**3** and **9**], one can see that if $2d/(d+1) < p \leq \infty$ and if $f \in L^p(X, \omega)$ is zonal about some $x_0 \in X$ then $\sum_{n=0}^{\infty} Y_n(f\colon x)$ converges almost everywhere on X. However, Theorem 1 shows that for $p = 2d/(d+1)$ the following behaviour occurs:

8. COROLLARY. *Let X be a compact two-point homogeneous space of dimension $d > 1$ and fix $x_0 \in X$. Then for $p = 2d/(d+1)$ there exists $f \in L^p(X, \omega)$ such that f is zonal about x_0, $f = 0$ a.e. on $\{y \in X: \rho(y, x_0) > l/2\}$ and $\sum_{n=0}^{\infty} Y_n(f: x)$ diverges almost everywhere on X.*

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