APPROXIMATING THE ABSOLUTELY CONTINUOUS MEASURES INVARIANT UNDER GENERAL MAPS OF THE INTERVAL

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Abstract. Let $\tau : I \to I$ be a nonsingular, piecewise continuous transformation which admits a unique absolutely continuous invariant measure $\mu$ with density function $f^\ast$. The main result establishes the fact that $f^\ast$ can be approximated weakly by the density functions of a sequence of measures invariant under piecewise linear Markov maps $\{\tau_n\}$ which approach $\tau$ uniformly.

1. Introduction. Let $\tau$ be a nonsingular, measurable transformation from $I = [0,1]$ into itself and let $\mathcal{B}$ denote the Lebesgue measurable subsets of $I$. A measure $\mu$ defined on $(I, \mathcal{B})$ is absolutely continuous if there exists a function $f : I \to [0, \infty)$, which is integrable with respect to Lebesgue measure $m$, i.e., $f \in \mathcal{L}^1(I, \mathcal{B}, m) \equiv \mathcal{L}_1$, and for which

$$\mu(S) = \int_S f(x) \, m(dx) \quad \forall S \in \mathcal{B}.$$ 

The measure $\mu$ is said to be invariant (under $\tau$) if $\mu(\tau^{-1}S) = \mu(S)$ for all sets $S \in \mathcal{B}$.

The Frobenius-Perron operator $P_\tau : \mathcal{L}_1 \to \mathcal{L}_1$ has proven to be a useful tool in the study of absolutely continuous invariant measures [1, 2]. It is defined by

$$(P_\tau f)(x) = \frac{d}{dx} \int_{\tau^{-1}[0, x]} f(s) \, m(ds).$$

The importance of $P_\tau$ lies in the fact that each of its fixed points is the density of a measure invariant under $\tau$, i.e., if $P_\tau f^\ast = f^\ast$, then

$$\mu(\cdot) = \int f^\ast(x) \, m(dx)$$

is invariant under $\tau$ [1].

In [2] a sequence of matrices $\{P_n\}$, depending on $\tau$, is constructed and the following result obtained:

**Theorem 1.** Let $\tau : I \to I$ be a piecewise $C^2$ map with $\inf |\tau'| > 2$. If $P_\tau$ has a unique fixed point $f^\ast$, then the sequence $\{f_n\}$ of fixed points (regarded as functions on $I$) of $\{P_n\}$ converges to $f^\ast$ in the $\mathcal{L}_1$-norm.
The proof of Theorem 1 depends on the fact that $P_\tau$, where $\tau$ is expanding, reduces the variation of the function on which it acts [1]. The critical inequality is
\[ \int_0^1 V_0 P_\tau f \leq \alpha \| f \| + \beta \int_0^1 f, \]
where $V_0^1$ denotes the variation on $[0, 1]$ and $\| \| \|$ the $\ell^1$-norm. In the proof of Theorem 1, $\beta$ must be less than 1 and this is truly only when $\inf |\tau'| > 2$.

When $\tau$ is nonexpanding, as for example if $\tau(x) = \gamma x(1 - x)$, where $\gamma$ can take on any value between 0 and 4, there are no known results similar to Theorem 1. The technique of [2] fails because $P_\tau f$ can have infinite variation for $f$ of bounded variation.

In this note we shall obtain a result analogous to Theorem 1 for a large class of transformations $\tau$ which admit a unique absolutely continuous invariant measure. To do this we shall use the weak topology on the space of measures and the result will be of the form: $f_n \overset{\omega}{\to} f^*$, where $\omega$ denotes weak convergence. Although this may not appear to be a strong result, it is sufficient for most statistical purposes; for example, all the moments of the density $f_n$ will be close to the corresponding ones for $f^*$, and
\[ \mu_n(S) = \int_S f_n(x) m(dx) \to \int_S f^*(x) m(dx) \equiv \mu(S) \]
for any $S \in \mathcal{B}$.

2. Notation. A piecewise continuous map $\tau_n: I \to I$ is called Markov if there exist points $a_0 < a_1 < \cdots < a_{n-1} < a_n$ such that for $i = 0, 1, \ldots, n - 1$, $\tau |_{I_i}$, where $I_i = (a_{i-1}, a_i)$, is a homeomorphism onto some interval $(\alpha_{i-1}, \alpha_i)$. The partition $J_n = \{I^n_i\}_{i=1}^n$ is referred to as a Markov partition with respect to $\tau$.

Now let $\tau: I \to I$ be piecewise continuous and nonsingular. Partition $I \times I$ into an $n \times n$ grid and form the piecewise linear map $\tau_n$ by joining corner points of the grid in such a way that $\tau_n$ approximates $\tau$. Clearly $\tau_n$ will have only integer slopes and $\tau_n \to \tau$ as $n \to \infty$ in the uniform norm.

The Frobenius-Perron operator $P_\tau$, when restricted to step functions on $J_n$, can be represented by a matrix [6], which we denote by $P_n$, its entries are given by
\[ p_{ij}^n = 1/\tau_{ij}^{n} \] if $I^n_i \subset \tau_n(I^n_j)$,
\[ = 0 \] otherwise.

In [3] it is shown that $P_n$ is similar to a stochastic matrix and therefore has a fixed point $f_n$, which we regard as a step function on $J_n$. Our aim is to prove that $f_n$ converges weakly to $f^*$, the density of the unique measure invariant under $\tau$.

**Definition 1.** Let $\{\mu_n\}$ be a sequence of absolutely continuous probability measure on $(I, \mathcal{B})$ and let $f_n$ be the density of $\mu_n$. We shall say that $f_n(\mu_n)$ converges weakly to the density $f$ (measure $\mu$) if and only if for each $g \in C$, the space of real, bounded and continuous functions on $I$,
\[ \int_I g(x)f_n(x)m(dx) \to \int_I g(x)f(x)m(dx) \]
as $n \to \infty$. 

In fact it is sufficient that $g$ is in a space $D$ dense in $C$ [4, Theorem 12.2]. For our purposes, we shall use $C^1$, the space of functions on $I$ which have continuous first derivatives.

**Definition 2.** Let $g$ be any function from $I$ into $(-\infty, \infty)$, and let $\delta$ and $\varepsilon$ be positive numbers. We denote by $\partial_{\delta, \varepsilon}(g)$ the set of those points $x \in I$ for which the distance between $g(x')$ and $g(x'')$ exceeds $\varepsilon$ for some pair of points $x'$, $x''$ in the open interval $(x - \delta, x + \delta)$.

A more general version of the following theorem is proved in [5].

**Theorem 2.** Let $\{g_n\}_{n \geq 1}$ be a sequence of bounded, real-valued and measurable functions defined on $S$ and let $\alpha$ be a real number. Then a necessary and sufficient condition that $\int g_n(x)f_n(x) \, m(dx) \to \alpha$ for every sequence $\{f_n\}$ converging weakly to $f$ is that

(i) $\{g_n\}_{n \geq 1}$ is uniformly bounded,
(ii) $\int g_n(x)f(x) \, m(dx) \to \alpha$ and
(iii) $\forall \varepsilon > 0$, $\lim_{\delta \to 0} \limsup_{n \to \infty} \int_{\partial_{\delta, \varepsilon}(g_n)} f(x) \, m(dx) = 0$.

It can be shown that (iii) holds iff

(iii') $\forall \varepsilon > 0$, for every sequence $\{\delta_k\}$ of positive numbers converging to 0, and for every subsequence $\{g_{n_k}\}$, $\int \cap_{k \geq 1} \partial_{\delta, \varepsilon}(g_{n_k}) f(x) \, m(dx) = 0$.

**Lemma 1.** Let $g$ be a bounded, piecewise continuous function on $[0,1]$ whose set of discontinuity points, $D$, has Lebesgue measure 0. Let $\{g_n\}$ be a uniformly bounded sequence of piecewise continuous functions which approaches $g$ uniformly. Then, if $f_n \to f$,

$$\int g_n(x)f_n(x) \, m(dx) \to \int g(x)f(x) \, m(dx)$$

as $n \to \infty$.

**Proof.** Since $g_n \to g$ uniformly, we have

$$\int g_n(x)f(x) \, m(dx) \to \int g(x)f(x) \, m(dx) = \alpha.$$

It remains to prove (iii'). Let $\varepsilon > 0$. Then for any sequence $\{\delta_k\}$ of positive numbers converging to 0 and every subsequence $\{g_{n_k}\}$, $\cap_{k=1}^{\infty} \partial_{\delta, \varepsilon}(g_{n_k}) \subset D$. Since $m(D) = 0$, (iii') is valid and Theorem 2 can be invoked. Q.E.D.

3. Main result.

**Lemma 2.** Let $\{\tau_n\}$ be a sequence of nonsingular transformations from $I \to I$ which approach $\tau$ uniformly. Let $f \in C^1$. Then

$$\int h(x)(P_{\tau_n}f)(x) \, m(dx) \to \int h(x)(P_{\tau}f)(x) \, m(dx)$$

as $n \to \infty$ for any $h \in C^1$. 

PROOF. From the definition of the Frobenius-Perron operator, we have
\[
\int f_h(x) (P_{\tau_n} f(x) - P_\tau f(x)) \, m(dx)
= \int f_h(x) \left\{ \frac{d}{dx} \int_{\tau_n^{-1}[0,x]} f(y) \, m(dy) - \frac{d}{dx} \int_{\tau^{-1}[0,x]} f(y) \, m(dy) \right\} m(dx).
\]
Integrating by parts,
\[
\int f_h(x) \left\{ \frac{d}{dx} \int_{\tau_n^{-1}[0,x]} f(y) \, m(dy) \right\} m(dx)
= \int f_h(x) \left\{ \frac{d}{dx} \int_{\tau^{-1}[0,x]} f(y) \, m(dy) \right\} m(dx).
\]
Thus,
\[
\int f_h(x) (P_{\tau_n} f(x) - P_\tau f(x)) \, m(dx)
= \int \left\{ \int_{\tau_n^{-1}[0,x]} f(y) \, m(dy) - \int_{\tau^{-1}[0,x]} f(y) \, m(dy) \right\} h'(x) \, m(dx).
\]
and
\[
\left| \int f_h(x) (P_{\tau_n} f(x) - P_\tau f(x)) \, m(dx) \right|
\leq \int \left| \int_{\tau_n^{-1}[0,x] \triangle (\tau^{-1}[0,x])} f(y) \, m(dy) h'(x) \, m(dx) \right|
\]
where \( \triangle \) denotes the symmetric difference. Since \( \tau_n \to \tau \) uniformly as \( n \to \infty \), \( m((\tau_n^{-1}[0,x] \triangle (\tau_n^{-1}[0,x])) \to 0 \) as \( n \to \infty \). Since \( h'(x) \) is continuous on \( I \), it is bounded. This completes the proof. Q.E.D.

We can now state the main result of this note.

Theorem 3. Let \( \tau: I \to I \) be a nonsingular, piecewise continuous map, whose set of discontinuities has Lebesgue measure 0, and let \( \tau \) admit a unique absolutely continuous probability measure \( \mu \). Let \( \{\tau_n\} \) be a sequence of piecewise linear Markov maps which approach \( \tau \) uniformly. Let \( f_n \) denote a fixed point of \( P_{\tau_n} \equiv P_{\tau_n}, \) where \( \|f_n\| = 1 \) and \( f_n > 0 \). Then \( f_n \to f^* \) as \( n \to \infty \), where \( f^* \) is the density function of \( \mu \).

PROOF. Since \( I \) is compact, the family of probability measures \( \{\mu_n\} \), defined by \( \mu_n(E) = \int_E f_n(x) \, m(dx) \), is weakly compact. Hence there exists a subsequence \( \{f_{n_k}\} \) and a function \( f: I \to I \) such that \( f_{n_k} \to f \).

Now, for any \( h \in C^1 \),
\[
\left| \int f_h(x) (f(x) - P_{\tau_n} f(x)) \, m(dx) \right|
\leq \left| \int f_h(x) (f(x) - f_{n_k}(x)) \, m(dx) \right| + \left| \int f_h(x) (f_{n_k}(x) - P_{\tau_n} f_{n_k}(x)) \, m(dx) \right|
+ \left| \int f_h(x) (P_{\tau_n} f_{n_k}(x) - P_{\tau_n} f(x)) \, m(dx) \right| + \left| \int f_h(x) (P_{\tau_n} f(x) - P_{\tau_n} f(x)) \, m(dx) \right|.
\]
The first term approaches 0 since \( f_n \overset{\omega}{\to} f \). Since \( P_n f_n = f_n \), the second term is identically 0. The fourth term approaches 0 by virtue of Lemma 2. Consider now the third term,

\[
\int_0^1 h(x) \frac{d}{dx} \left( \int_{\tau_n^{-1}[0, x]} \left( f_n(y) - f(y) \right) m(dy) \right) m(dx)
\]

\[
= \int_0^1 \left( \int_{\tau_n^{-1}[0, x]} \left( f_n(y) - f(y) \right) m(dy) \right) h'(x) m(dx).
\]

Fix \( x \in [0, 1] \) and consider

\[
A_n(x) = \int_{\tau_n^{-1}[0, x]} \left( f_n(y) - f(y) \right) m(dy)
\]

\[
= \int_{\tau_n^{-1}[0, x]} f_n(y) m(dy) - \int_{\tau_n^{-1}[0, x]} f(y) m(dy).
\]

Now \( \chi_{\tau_n^{-1}[0, x]} \) is a piecewise continuous step function which approaches \( \chi_{\tau^{-1}[0, x]} \) uniformly as \( n \to \infty \). Clearly

\[
\int_{\tau_n^{-1}[0, x]} f(y) m(dy) - \int_{\tau_n^{-1}[0, x]} f(y) m(dy) \equiv 0
\]

as \( n \to \infty \). Thus, it follows from Lemma 1 that \( A_n(x) \to 0 \) as \( n \to \infty \). Note that \( |A_n(x)| \leq 2 \). Since \( h \in C^1 \), \( |h'(x)| \leq L < \infty \). Hence, the Dominated Convergence Theorem implies that

\[
\int_0^1 A_n(x) h'(x) m(dx) \to 0
\]

as \( n \to \infty \). We have, therefore, established, for any \( h \in C^1 \),

\[
\int_I h(x)(f(x) - P_n(x)) m(dx) = 0.
\]

This means \( P_n f(x) = f(x) \) m.a.e. But \( f^* \) is the unique fixed point of \( P \). Thus \( f = f^* \) m.a.e., and \( f_n \overset{\omega}{\to} f^* \). We have therefore shown that any weakly convergent subsequence of \( \{f_n\} \) converges weakly to \( f \). Hence \( f_n \overset{\omega}{\to} f \) as \( n \to \infty \). Q.E.D.

**Remarks.** (1) Theorem 1 establishes a necessary condition for the existence of an absolutely continuous invariant for a general map \( \tau: I \to I \).

(2) Classes of maps \( \tau: I \to I \) which are nonexpanding and which have unique absolutely continuous invariant measures can be found in [7–10]. Theorem II.8.3 of [10] describes some of the results in [7].

**References**


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