RECOVERABILITY OF SOME CLASSES OF ANALYTIC FUNCTIONS FROM THEIR BOUNDARY VALUES*

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Abstract. The technique devised by D. J. Patil to recover the functions of the Hardy space $H^p (1 \leq p \leq \infty)$ from the restrictions of their boundary values to a set of positive measure on the unit circle was modified by S. E. Zarantonello in order to extend the result to $H^p (0 < p < 1)$.

In this paper, we show that Zarantonello's technique can be slightly modified to extend the result to a larger class of analytic functions in the unit disc. In particular, if $f(z)$ is analytic in the unit disc and satisfies

$$\lim_{r \to 1} (1 - r)^\beta \log M(r, f) = 0 \quad \text{for some} \quad \beta > 1,$$

then $f(z)$ can be recovered from the restriction of its boundary value to an open arc.

1. Introduction. Let $\mathbb{D}$ denote the open unit disc, $\partial \mathbb{D}$ its boundary, i.e., the unit circle, and $\mu$ the normalized Lebesgue measure on $\partial \mathbb{D}$. Furthermore, let $H^p (0 < p \leq \infty)$ denote the Hardy class of analytic functions in $\mathbb{D}$. It is well known that if $f(z) \in H^p$ and $\lim_{r \to 1} f(re^{i\theta}) = 0$ on a set $E \subset \partial \mathbb{D}$ of positive measure, then $f(z)$ is identically zero. We call this property the uniqueness property. In a sense, this means that functions in $H^p$ are uniquely determined by their values on $E$.

The question of whether a function $f \in H^p$ can be recovered from its restriction to $E$ was answered in the affirmative by D. Patil [1] for $1 \leq p \leq \infty$, who constructed a sequence of analytic functions that converged to $f$ uniformly on compact subsets of $\mathbb{D}$ as well as in the norm. Modifying Patil's technique, S. Zarantonello [4] and G. Walker [3] independently were able to extend the results to $0 < p < 1$ but in a slightly more restrictive form.

A function $f(z) \in H^p (0 < p < 1)$ can be recovered from the restriction of its distributional boundary value to $E$ where $E$ now is an open arc in $\partial \mathbb{D}$.

Since there are larger classes of analytic functions in $\mathbb{D}$ having the uniqueness property, e.g., the Nevanlinna class $N$, a natural question immediately arises, can functions in these classes be recovered from their restrictions to $E$? The purpose of this article is to show that the answer is "yes" for a large class of analytic functions. More precisely, if we denote by $\mathcal{H}(\alpha) (0 < \alpha < 1)$ the class of all analytic functions in $\mathbb{D}$ having the property $\int_0^1 M(r, f)\exp(-c/(1 - r)^\beta) \, dr < \infty$ for all $c > 0$ where

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\[ M(r, f) = \sup_{|z|=r} |f(z)| \] and \( \beta = \alpha/(1 - \alpha) \), then any function \( f(z) \in \mathcal{K}(1) = \bigcup_{0 < \alpha < 1} \mathcal{K}(\alpha) \) can be recovered from the restriction of its distributional boundary value to an open arc \( E \). The technique we use is a modification of the ones given by Patil and Zarantonello.

2. Preliminaries. For \( 0 < \alpha < 1 \), let \( \mathcal{K}(\alpha) \) be the space of all holomorphic functions \( F(z) \) in \( \mathbb{D} \) such that \( F(z) = \sum_{n=0}^{\infty} a_n z^n \) with \( a_n = O(e^{\alpha(n)}) \) as \( n \to \infty \). The topology of \( \mathcal{K}(\alpha) \) is the topology induced by the seminorms

\[
\| F \|_c = \sum_{n=0}^{\infty} |a_n| e^{-c \alpha n} < \infty \quad \text{for } c > 0.
\]

In [5 and 6], we showed the following facts.

1. \( F(z) \in \mathcal{K}(\alpha) \) if and only if

\[
\int_{0}^{1} M(r, f) \exp \left[ -\frac{c}{(1 - r)^\beta} \right] \, dr < \infty \quad \text{for all } c > 0.
\]

Moreover, the two families of seminorms \( \| \cdot \|_c \) and \( \| \cdot \|_c \) are equivalent.

2. Provided with that topology, \( \mathcal{K}(\alpha) \) becomes a Fréchet-Montel space whose topology is stronger than the topology of uniform convergence on compact subsets of \( \mathbb{D} \).

3. If we denote by \( \mathcal{D}_\alpha(\mathbb{D}) \) the algebra of functions \( G(z) \) that are analytic in \( \mathbb{D} \) and continuous in \( \mathbb{D} \) such that \( G(z) = \sum_{n=0}^{\infty} b_n z^n \), \( b_n = O(e^{-cn}) \) as \( n \to \infty \) for some \( c > 0 \), then for any \( \phi \in \mathcal{K}^*(\alpha) \) (the dual space of \( \mathcal{K}(\alpha) \)), there exists a unique function \( G(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathcal{D}_\alpha(\mathbb{D}) \) such that

\[
\phi(F) = \lim_{r \to 1} \int_{\partial \mathbb{D}} F(re^{i\theta}) G(e^{-i\theta}) \, d\mu = \sum_{n=0}^{\infty} a_n b_n.
\]

Conversely, any \( G(z) \in \mathcal{D}_\alpha(\mathbb{D}) \) defines a continuous linear functional on \( \mathcal{K}(\alpha) \) via (2.2).

Let \( \mathcal{D}_\alpha = \mathcal{D}_\alpha(\partial \mathbb{D}) \) be the space of all \( C^\infty \)-functions \( \phi \) on \( \partial \mathbb{D} \) such that

\[
\phi(\theta) = \sum_{-\infty}^{\infty} b_n e^{i\theta n}
\]

with \( b_n = O(e^{-c|n|^\alpha}) \) as \( n \to \infty \) for some \( c > 0 \).

Another useful characterization of the class \( \mathcal{D}_\alpha \) is that \( \phi \in \mathcal{D}_\alpha \) if and only if

\[
\sup_{0 < \theta < 2\pi} |\phi^n(\theta)| \leq KR^{n/\alpha} n^{\alpha/\alpha} \text{ for some constants } K \text{ and } R.
\]

Indeed, the class \( \mathcal{D}_\alpha \) is the same as the class \( C((n!)^{1/\alpha}) \) in the notation of [2].

It follows from the Denjoy-Carleman theorem that \( \mathcal{D}_\alpha \) is a nonquasianalytic class of functions on \( \partial \mathbb{D} \). A topology is defined on \( \mathcal{D}_\alpha \) by means of the fundamental system of neighborhoods of the origin

\[
V(\lambda) = \left\{ \phi \mid \phi(\theta) = \sum_{-\infty}^{\infty} b_n e^{i\theta n}, \quad b_n = O(e^{-\lambda |n|^\alpha}) \right\}
\]

where \( \lambda = (\lambda_n)_{n=0}^{\infty} \) and \( \lambda_n \neq 0 \).

The strong dual \( \mathcal{D}_\alpha^* \) of \( \mathcal{D}_\alpha \) is a space of Beurling distributions which are more general than Schwartz distributions (see [5 and 6] for references). One can verify the following.
1. Every Beurling distribution $f$ has a Fourier series expansion $f = \sum_{n=0}^{\infty} a_n w^n$ that converges weakly to it, where $w = e^{i\theta}$ and $a_n = \langle f, w^n \rangle$.

2. A necessary and sufficient condition that $f = \sum_{n=0}^{\infty} a_n w^n \in \mathcal{A}(\alpha)$ is that $a_n = O(e^{\alpha n |n|})$ as $n \to \infty$. Furthermore, if $\psi = \sum_{n=0}^{\infty} b_n w^n \in \mathcal{A}(\alpha)$, then $\langle f, \psi \rangle = \sum_{n=0}^{\infty} a_n b_n$.

3. If $F(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{K}(\alpha)$, then $f = \sum_{n=0}^{\infty} a_n w^n$ is a Beurling distribution belonging to the space $\mathcal{A}(\alpha)$. In addition to that (cf. [5, Corollary 3.1])

$$F(w) = F(rw) \quad \text{in } \mathcal{A}(\alpha).$$

As in [4], we say that $f$ is the distributional boundary value of $F(z)$ and $F(z)$ is the holomorphic extension of $f$. The space of all distributional boundary values of functions in $\mathcal{K}(\alpha)$ will be denoted by $\mathcal{A}(\alpha)$. We provide $\mathcal{A}(\alpha)$ with a topological structure isometric to that of $\mathcal{K}(\alpha)$ by setting

$$\|f\|_\alpha = \|F\|_\alpha.$$

From Cauchy's formula

$$F(rz) = \int_{\partial \mathcal{K}} \frac{F(w)}{1 - wz} d\mu(w)$$

and equation (2.3), it follows that $F(z) = \langle f, C_z \rangle$ where $C_z(w) = 1/(1 - wz)$.

3. Toeplitz operators on $\mathcal{A}(\alpha)$. Since $\mathcal{A}(\alpha)$ is an algebra [2], we define the multiplication operator $M_\phi$ for $\phi \in \mathcal{A}(\alpha)$ by

$$\langle M_\phi f, \psi \rangle = \langle f, \phi \psi \rangle$$

where $f \in \mathcal{A}(\alpha)$ and $\psi \in \mathcal{A}(\alpha)$.

The projection operator $P: \mathcal{A}(\alpha) \to \mathcal{A}(\alpha)$ is formally given by

$$P \left( \sum_{n=0}^{\infty} a_n w^n \right) = \sum_{n=0}^{\infty} a_n w^n$$

and hence the Toeplitz operator $S_\phi$ can now be defined on $\mathcal{A}(\alpha)$ as $S_\phi = PM_\phi$ for $\phi \in \mathcal{A}(\alpha)$.

**Lemma 1.** Let $\phi \in \mathcal{A}(\alpha)$ and $f \in \mathcal{A}(\alpha)$, then:

(i) For any $c > 0$ there exists $\tilde{c}$ such that

$$\|S_\phi f\|_\alpha \leq K(\phi, c) \|f\|_\alpha$$

where $K(\phi, c)$ is a constant that depends only on $c$ and $\phi$ but not on $f$.

(ii) $S_\phi f$ is the distributional boundary value of the analytic function $\langle M_\phi f, C_z \rangle$.

**Proof.** (i) Let $f = \sum_{n=0}^{\infty} a_n w^n$ and $\phi = \sum_{n=0}^{\infty} b_n w^n$, then $S_\phi f = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} a_n b_k w^k$.

Hence,

$$\|S_\phi f\|_\alpha \leq \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} |a_n| |b_k - n| e^{-c|k|n} = \sum_{n=0}^{\infty} |a_n| \sum_{k=-n}^{\infty} |b_k| e^{-c|k+n|n} = I_1 + I_2.$$
But \((k + n)^a \leq k^a + n^a \leq 2(k + n)^a\), therefore

\[
(3.2) \quad I_2 \leq \sum_{n=0}^{\infty} |a_n| \sum_{k=0}^{\infty} |b_k| \exp \left( -\frac{1}{2} c |k|^a - \frac{1}{2} c |n|^a \right) \leq B \| f \|_{c/2}
\]

where \(B = \sum_{k=0}^{\infty} |b_k| \exp(-\frac{1}{2} c |k|^a)\) which certainly converges since \(|b_k| \leq Ae^{-\frac{|k|^a}}\) for some \(b > 0\). On the other hand,

\[
(3.3) \quad I_1 \leq \sum_{n=0}^{\infty} |a_n| \sum_{k=-n}^{-1} Ae^{\frac{|k|^a - c|k+n|^a}}
\]

where \(d = \frac{1}{2} \min(b, c)\) and \(\hat{B} = \max_{0 \leq n} A(n/e^{dn^a}) < \infty\).

From (3.1), (3.2) and (3.3), it follows that \(\| S_{\phi} f \|_{c} \leq K \| f \|_{c}\) where \(K = \max(B, \hat{B})\) and \(\hat{c} = \min(c/2, d)\).

(ii) Let \( f \in \mathcal{H}(\alpha)\) and \( F(z) \) be its holomorphic extension to \(\partial D\). From Corollaries 3.1 and 3.2 of [5], it follows that \( F_r(w) = F(rw) \to f \) in \(\mathcal{H}(\alpha)\) and that \( H^2 \) is dense in \(\mathcal{H}(\alpha)\). Moreover, \( F_r(w) \) is the boundary value of a function in \( H^2 \), namely \( F_r(z) \).

Now an argument similar to that of Theorem 3.2 of [4] finishes the proof.

4. Recoverability Theorem. In this section we shall use the same notation as [4]. Let \( E \) be an open arc in \( \partial D \) and \( \psi \in \mathcal{C}_a \) such that \( 0 = \psi \leq 1 \), supp \( \psi \subset E \) and \( \mu(J) > 0 \) where \( J = \{ w \in \partial D : \psi(w) = 1 \} \). Such a \( \psi \) certainly exists since the class \( \mathcal{C}_a \) is nonquasianalytic. For each \( 0 < \lambda < 1 \) define

\[
\phi_{\lambda}(w) = \frac{1}{1 - \lambda \psi(w)}
\]

and

\[
H_{\lambda}(z) = \exp \left\{ -\frac{1}{2} \int_{\partial D} \frac{w \cdot z}{w - z} \log \phi_{\lambda}(w) \, d\mu \right\}, \quad z \in \partial D.
\]

As is shown in [1 and 4], if we denote the boundary value of \( H_{\lambda}(z) \) by \( h_{\lambda}(w) \), we have

(a) \(| h_{\lambda}(w) |^{-2} = \phi_{\lambda}(w) \).

(b) \( h_{\lambda} \) and \( h_{\lambda}^{-1} \) are in \( H^\infty(\partial D) \).

**Lemma 2.** (i) \( \phi_{\lambda} \in \mathcal{C}_a \).

(ii) \( \mathcal{S}_{\phi_{\lambda}} : \mathcal{C}_a^* \to \mathcal{C}_a^* \) is invertible with inverse \( \mathcal{S}_{h_{\lambda}}^{-1} = \mathcal{S}_{h_{\lambda}} \mathcal{S}_{h_{\lambda}^{-1}} \).

(iii) For any \( c > 0 \), there exists \( \hat{c} \) such that

\[
\| \mathcal{S}_{\phi_{\lambda}} f \|_{c} \leq K \| f \|_{\hat{c}} \quad \text{where } K \text{ is independent of } \lambda.
\]

**Proof.** (i) Since \( \psi \in \mathcal{C}_a \) and \( \mathcal{C}_a \) is an inverse-closed, nonquasianalytic class of functions [2], it follows immediately that \( \phi_{\lambda} \in \mathcal{C}_a \).

(ii) It suffices to show that \( h_{\lambda} \in \mathcal{C}_a \). Let \( h_{\lambda}(w) = e^{-(u(w) + i v(w))} \). Since \( \log \phi_{\lambda}(w) \in \mathcal{C}_a \) by part (i) and Theorem A of [2], it is easy to see that \( u(w) \) and \( v(w) \) are also in \( \mathcal{C}_a \). We invoke Theorem A once more to show that \( h_{\lambda} \) is indeed in \( \mathcal{C}_a \).

(iii) \( \| \mathcal{S}_{h_{\lambda}} f \|_{c} \leq \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\hat{h}_{\lambda}(k)||\hat{f}(m)||\hat{f}(n)|e^{-c|k-m+n|^a} \).
Since $|\hat{h}(w)| \leq (\phi(w))^{-1/2} \leq 1$, it follows that the sequence $\{\hat{h}(k)\}$ is uniformly bounded. Now an argument similar to the one given in Lemma 1 yields the result.

**Lemma 3.** For fixed $z \in \partial$ and $c > 0$, $\lim_{\lambda \to 1} \|S_{\phi}^{-1}C \|_c = 0$.

**Proof.** The relation (cf. [4, equation (4.4.2)])

$$|H_\lambda(z)| = \exp \left\{ -\frac{1}{2} \int \frac{1}{1 - wz} \log \phi(w) \, d\mu(w) \right\} \leq (1 - \lambda)^\alpha$$

where $2\alpha = \{(1 - |z|)/(1 + |z|)\} \mu(J) > 0$, shows that $H_\lambda(z) \to 0$ uniformly on compact subsets of $\partial$ as $\lambda \to 1$. Since $H_\lambda(z) = \sum_{k=0}^\infty \hat{h}(k)z^k$, it is routine to show that $\hat{h}(k) \to 0$ as $\lambda \to 1$ for $k = 0, 1, 2, \ldots$. Moreover, $|\hat{h}(k)| \leq B$ for all $0 \leq \lambda < 1$ and $k = 0, 1, 2, \ldots$.

We claim that $H_\lambda(z) \to 0$ in $\mathcal{C}(\alpha)$ as $\lambda \to 1$. For

$$\lim_{\lambda \to 1} \|S_{\phi}^{-1}C \|_c = \lim_{\lambda \to 1} \|H_\lambda(z)h_\lambda C \|_c = 0$$

for some $c > 0$ and a constant $A$ that depends on $z$ and $c$ but not on $\lambda$.

**Lemma 4.** Let $f \in \mathcal{C}_a^*$ and $g_\lambda = S_{\phi}^{-1}(S_{\phi} - 1)f$, then $\lim_{\lambda \to 1} g_\lambda = f$ in $\mathcal{C}_a^*$.

**Proof.** As a consequence of Corollary 3.2 of [5], $L^\infty(\partial \Omega)$ is dense in $\mathcal{C}_a^*$. From this and the fact that $\{C_z : z \in \partial\}$ is a fundamental set in $L^2(\partial \Omega)$ and that the embedding $L^2(\partial \Omega) \to \mathcal{C}_a^*$ is continuous, we conclude that $\{C_z : z \in \partial\}$ is also a fundamental set in $\mathcal{C}_a^*$. This fact together with Lemmas 2 and 3 may now be used to show that $\lim_{\lambda \to 1} S_{\phi}^{-1}f \|_c = 0$. But $S_{\phi}^{-1}f = f - g_\lambda$ which yields the result.

Using arguments parallel to those given in [4 and 1], the reader should be able to finish the proof of the following theorem.

**Theorem.** Let $E$ be an open arc of $\partial \Omega$. Let $F(z) \in \mathcal{C}(\alpha)$ and $f$ be its distributional boundary value on $\partial \Omega$. Suppose that $g$ is the restriction of $f$ to $E$. For each $0 < \lambda < 1$ define an analytic function $G_\lambda(z)$ on $\Omega$ by

$$G_\lambda(z) = H_\lambda(z) \langle g \cdot (\phi - 1)h_\lambda C \rangle.$$

Then, $\lim_{\lambda \to 1} G_\lambda(z) = F(z)$ in $\mathcal{C}(\alpha)$. In particular, $\lim_{\lambda \to 1} G(z) = F(z)$ uniformly on compact subsets of $\Omega$.

It is well known that functions in $\mathcal{C}(\alpha)$ can have radial limits equal to zero on a set $E$ of positive measure (in fact, even on a set of measure $2\pi$) without being
identically zero i.e., the class $\mathcal{Y}(\alpha)$ does not have the uniqueness property. However, the class $\mathcal{Y}(\alpha)$ possesses another uniqueness property which we state as a corollary.

**Corollary.** Let $F(z) \in \mathcal{Y}(\alpha)$ and $f$ be its distributional boundary value. If $f = 0$ (in the sense of distributions) on an open arc $E$, then $F(z)$ is identically zero.

**Remarks.** (i) One may ask how far can the result be extended? To answer this question, let us consider the class $\mathcal{Y}(\omega)$ of all analytic functions $F(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathbb{D}$ with $a_n = \mathcal{O}(e^{\omega(n)})$ as $n \to \infty$ where $\omega(x)$ is continuously differentiable and monotonically increasing on $[0, \infty)$. Certainly, our technique fails if $\sum_{n=0}^{\infty} \omega(n)/n^2 = \infty$, since in this case the class $\mathcal{C}_\omega$ is quasianalytic by the Denjoy-Carleman theorem. Therefore, if the function $\psi(w) = 0$ on a set of positive measure, it is identically zero on $\partial \mathbb{D}$ and hence $G_\lambda(z)$ is identically zero for all $0 < \lambda < 1$.

(ii) The class $N^+$ provided with the topology induced by the metric

$$(f, g) = \int_{\partial \mathbb{D}} \log(1 + |f(e^{i\theta}) - g(e^{i\theta})|) d \mu$$

where $f$ and $g \in N^+$ is a subspace of $\mathcal{Y}(1/2)$.

From the main theorem, it follows that if $F \in N^+$, then $G_\lambda(z) \to F(z)$ in $\mathcal{Y}(1/2)$. It would be interesting to know whether $G_\lambda(z) \to F(z)$ in $N^+$.

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**References**


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