TOTALLY ANALYTIC SPACES UNDER $V = L$

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Abstract. The following results obtain under the axiom of constructibility ($V = L$):

Assume that every subset of a topological space $X$ is analytic. Then $X$ is
\sigma-left-separated. Moreover, if the character of $X$ is $\leq \omega_1$, then $X$ is \sigma-discrete.

Assume that $X$ is a perfectly normal space of character $\leq \omega_1$ such that every
subset of $X$ belongs to the $\sigma$-algebra generated by the analytic subsets of $X$. Then $X$
is $\sigma$-discrete.

1. Introduction. By an analytic subset of a topological space $X$, we mean a set that
can be obtained from a family of closed subsets of $X$ by the $A$-operation, i.e., a set $S$ the form
$\bigcup_{\sigma \in \omega} \bigcap_{n \in \omega} F_{\sigma n}^n$, where the sets $F_s$, $s \in \omega$, are closed in $X$. We say that
$X$ is a totally analytic space if each subset of $X$ is analytic. This notion generalizes
that of a space in which each subset is an $A$-set.

A space $X$ is $\sigma$-discrete ($F_\sigma$-discrete) if $X$ is the countable union of discrete (and
closed) subspaces. $X$ is left-separated if there exists a well-order $<\!$ on $X$ such that all
the initial segments $\{y \in X: y < x\}$, $x \in X$, are closed in $X$. We say that $X$
is $\sigma$-left-separated if $X$ is the countable union of left-separated subspaces. Note that the
concept of an $F_\sigma$-left-separated space is superfluous, since such a space is actually
left-separated; in particular, every $F_\sigma$-discrete space is left-separated.

The character of a space $X$ is the least cardinal number $\kappa$ such that every point of
$X$ has a neighbourhood-base consisting of at most $\kappa$ sets; the character of $X$ is
denoted by $\chi(X)$.

The results in this paper give some partial answers to the question whether it is
consistent with ZFC that every totally analytic space is $\sigma$-discrete. We can only aim
for a consistency result since it follows from $MA + \neg CH$ that $R$ has uncountable
(hence non-$\sigma$-discrete) subspaces that are totally analytic (and in fact, with each
subset an $F_\sigma$-set). To obtain our results, we use Gödel's constructibility axiom
$V = L$. Previous to the results below, G. M. Reed has shown that, under $V = L$,
each first countable normal space with every subset an $F_\sigma$-set is $F_\sigma$-discrete [8], and
R. W. Hansell has shown that, under $V = L$, a totally analytic space $X$ of character
$\leq \omega_1$ is $\sigma$-discrete provided that the product of $X$ with the irrationals is normal [5].
Like Reed and Hansell, we rely on W. G. Fleissner's techniques and results in [2]
dealing with the seemingly unrelated topic of collectionwise Hausdorffness of
normal spaces of character $\leq \omega_1$ under $V = L$.

Received by the editors November 23, 1981 and, in revised form, May 25, 1982.
1980 Mathematics Subject Classification. Primary 54H05, 04A15, 28A05; Secondary 54A25, 54A35.
Key words and phrases. $V = L$, analytic set, $\sigma$-discrete, left-separated, character.
2. The main theorems. We have only been able to obtain $\sigma$-discreteness from $V = L$ for totally analytic spaces under the restriction that the spaces under consideration have character $\leq \omega_1$; for general spaces, we have the following result:

**Theorem 2.1** ($V = L$). Each totally analytic space is $\sigma$-left-separated.

The proof of Theorem 2.1 is a modification of Fleissner's proof of the main result in [2]. We shall give some lemmas from which the theorem follows; for a few of these lemmas, we do not give a proof but only indicate the changes that are needed in the proofs of the corresponding results in [2].

In [2], mappings $A \to \omega_1$ were used to code open separations of subsets of a discrete subset $A$ of a space $X$ (of character $\leq \omega_1$). Here we use mappings $X \to \omega_1$ to code partitions of $X$ into analytic sets. (Note that, under $V = L$, $|\omega_1| = \omega_1$.)

**Lemma 2.2.** Let $\mathcal{C}$ be a disjoint family of analytic subsets of $X$. Then there exists a function $f: X \to \omega_1$ such that, for each $C \in \mathcal{C}$,

$$C = \bigcup_{x \in C} \bigcap_{n \in \omega} \{x \in C : f(x) = n \}.$$

**Proof.** For every $C \in \mathcal{C}$, let $\{F_s(C) : s \in \omega_1\}$ be a family of closed subsets of $X$ such that $C = \bigcup_{s \in \omega_1} F_{f(s)(C)}$. Define $f: X \to \omega_1$ as follows. If $x \in X \setminus \bigcup \mathcal{C}$, let $f(x)$ be an arbitrary element of $\omega_1$. If $x \in C \in \mathcal{C}$, let $f(x) \in \omega_1$ be such that $x \in \bigcap_{n \in \omega} F_{f(x)(n)}(C)$. That $f$ has the required property follows by observing that for all $C \in \mathcal{C}$, $\phi \in \omega_1$, and $n \in \omega$, $\{x \in C : f(x) = n \} \subseteq F_{\phi(n)}(C)$. □

A connection between mappings $X \to \omega_1$ and $\sigma$-left-separatedness of $X$ is given in the following lemma. Call $X$ $A$-left-separated if there exists a well-order $\prec$ on $X$ and a map $f: X \to \omega_1$ such that, for each $x \in X$,

$$x \notin \bigcup_{\phi \in \omega_1} \bigcap_{n \in \omega} \{y \prec x : f(y) = n \}.$$

**Lemma 2.3.** (i) Every $A$-left-separated space is $\sigma$-left-separated.

(ii) If a space has a countable cover by analytic left-separated subspaces then the space is $A$-left-separated.

**Proof.** (i) Assume that $X$, $\prec$, and $f$ satisfy the condition defining $A$-left-separatedness. For each $x \in X$, let $n(x) \in \omega$ be such that $x \notin \{y < x : f(y) = f(x) \cap n(x)\}$. For each $s \in \omega_1$, let $L_s = \{x \in X : f(x) \cap n(x) = s\}$. Then the sets $L_s$, $s \in \omega_1$, cover $X$, and it is easy to check that these sets are left-separated as subspaces of $X$.

(ii) Assume that $X$ has a cover $\{A_k : k < \omega_1\}$ by analytic subsets and that, for every $k < \omega_1$, there exists a well-ordering $\prec_k$ of $A_k$ such that, for each $x \in A_k$, $x \notin \{y \in A_k : y <_k x\}$. For each $k < \omega_1$, let $\{F_{\phi}^k : \phi \in \omega_1\}$ be a family of closed subsets of $X$ such that $A_k = \bigcup_{\phi \in \omega_1} \bigcap_{n \in \omega} F_{\phi(n)}^k$ and, for all $\phi \in \omega_1$ and $n \in \omega$, $F_{\phi(n)}^{k+1} \subseteq F_{\phi(n)}^k$. For each $x \in X$, let $k(x)$ be the least $k \in \omega_1$ such that $x \in A_k$, and let $f(x) \in \omega_1$ be such that $x \in \bigcap_{n \in \omega} F_{f(x)(n)}^{k(x)}$. Define a well-order $\prec$ on $X$ by setting $y < x$ iff either $k(y) < k(x)$ or $k(y) = k(x)$ and $y <_k x$. To show that $X$ is $A$-left-separated, we need to show that $X$ is $\sigma$-left-separated.

For each $\phi \in \omega_1$, let $L_\phi = \{x \in X : f(x) = \phi \}$. Then the sets $L_\phi$, $\phi \in \omega_1$, cover $X$, and it is easy to check that these sets are left-separated as subspaces of $X$. □
A-left-separated with respect to \( f \) and \(<\), let \( x \in X \) and let \( k = k(x) \). For all \( \phi \in \omega \omega \) and \( n \in \omega \), we have \( \{ y < x : f(y) | n = \phi | n \} = \bigcup_{j < k} \{ y \in A_j : f(y) | n = \phi | n \} \cup \{ y \in A_k : y < k \; x \text{ and } f(y) | n = \phi | n \} \), and it follows that

\[
\{ y < x : f(y) | n = \phi | n \} \subseteq \bigcup_{j < k} F_{\phi | n}^j \cup \{ y \in A_k : y < k \; x \} \cap F_{\phi | n}^k.
\]

Further, since the sets \( F_{\phi | n}^i, \; n \in \omega \), are decreasing, it follows that

\[
\bigcap_{n \in \omega} \{ y < x : f(y) | n = \phi | n \} \subseteq \bigcup_{j < k} \left( \bigcap_{n \in \omega} F_{\phi | n}^j \right) \cup \{ y \in A_k : y < k \; x \} \cap \bigcap_{n \in \omega} F_{\phi | n}^k \quad \subseteq \bigcup_{j < k} A_j \cup \{ y \in A_k : y < k \; x \} \cap A_k
\]

\[
= \bigcup_{j < k} A_j \cup \{ y \in A_k : y < k \; x \} = \{ y \in X : y < x \}.
\]

Consequently, \( \bigcup_{\phi \in \omega} \bigcap_{n \in \omega} \{ y < x : f(y) | n = \phi | n \} \subseteq \{ y : y < x \} \); since \( x \notin \{ y : y < x \} \), we have completed the proof of (ii). \( \Box \)

Remark. By Lemma 2.3, a totally analytic space is \( \sigma \)-left-separated iff the space is \( A \)-left-separated. Thus we can prove Theorem 2.1 by showing that, under \( V = L \), each totally analytic space is \( A \)-left-separated. Note that, corresponding to the similarly marked result on p. 295 of [2], we have the following consequence of Lemma 2.3:

\((*)\) If a totally analytic space is the countable union of \( A \)-left-separated subspaces, then the space is \( A \)-left-separated.

For each cardinal number \( \kappa \), denote by \( \text{TALS}(\kappa) \) (\( \text{TALS}(<\kappa) \)) the statement that each totally analytic space of cardinality \( \kappa \) (of cardinality \( <\kappa \)) is \( A \)-left-separated. For \( \kappa \leq \omega \), \( \text{TALS}(\kappa) \) holds trivially. We shall prove Theorem 2.1 by using transfinite induction on \( \kappa \) to show that \( \text{TALS}(\kappa) \) holds for every uncountable cardinal \( \kappa \). For a singular cardinal \( \kappa \) of cofinality \( \omega \), it follows from \((*)\) that \( \text{TALS}(<\kappa) = \text{TALS}(\kappa) \). Hence it suffices to prove that \( \text{TALS}(<\kappa) = \text{TALS}(\kappa) \) whenever \( \kappa \) is either regular or singular of cofinality \( >\omega \).

Without loss of generality, we may assume that the underlying set of our totally analytic space \( X \) is a cardinal number \( \kappa \); we shall make this assumption for the rest of the proof of Theorem 2.1.

For all \( f : \kappa \to \omega \) and \( \alpha < \kappa \), let \( W_{\alpha}^f = \bigcup_{\phi \in \omega} \bigcap_{n \in \omega} \{ \beta < \alpha : f(\beta) | n = \phi | n \} \).

The following result corresponds to Lemma 1 of [2]:

**Lemma 2.4.** Assume \( \text{TALS}(<\kappa) \). Either \( A_f = \{ \alpha <\kappa : W_{\alpha}^f \neq \emptyset \} \) is stationary for each \( f : \kappa \to \omega \) or \( X \) is \( A \)-left-separated.

**Proof.** Assume that there exists \( f : \kappa \to \omega \) and a cub \( C \subseteq \kappa \) such that \( C \cap A_f = \emptyset \).

For each \( \alpha \in \kappa \), let \( \gamma(\alpha) \) be the least element of the set \( \{ \gamma \in C : \alpha < \gamma \} \). For every \( \gamma \in C \), it follows from \( \text{TALS}(<\kappa) \) that there exists a well-order \( <_\gamma \) on \( \gamma \) and a map \( f_\gamma : \gamma \to \omega \) such that for each \( \alpha \in \gamma \), \( \bigcup_{\phi \in \omega} \bigcap_{n \in \omega} \{ \beta < \alpha : f_\gamma(\beta) | n = \phi | n \} \cap \gamma = \{ \beta \in \gamma : \beta <_\gamma \alpha \} \). Define a well-order \( < \) on \( \kappa \) by setting \( \alpha < \beta \iff \text{either } \gamma(\alpha) < \gamma(\beta) \) or \( \gamma(\alpha) = \gamma(\beta) \) and \( \alpha < _{\gamma(\alpha)} \beta \). Define a binary operation \( + \) on \( \omega \) by the condition that \( \theta + \psi(2n) = \theta(n) + \psi(n) \) for all \( \theta, \psi \in \omega \) and \( n \in \omega \). Define
a map \( g: \kappa \rightarrow \omega \) by setting \( g(\alpha) = f \upharpoonright f_{\gamma}(\alpha) \) for each \( \alpha \in \kappa \). We show that \( X \) is \( A \)-left-separated with respect to \( < \) and \( g \). Let \( \alpha \in \kappa \), and let \( \gamma = \gamma(\alpha) \). Denote by \( \delta \) the largest element of the set \( \{ \beta \in \kappa \cup \{0\}: \rho \leq \alpha \} \), and let \( B = \{ \beta \in \gamma \sim \delta: \beta < \gamma, \alpha \} \); note that \( \{ \beta \in \kappa: \beta < \alpha \} = \delta \cup B \) and that, for each \( \beta \in B \), \( \gamma(\beta) = \gamma \). Let \( \phi \in \omega \). To show that \( \bigcap_{n \in \omega} \{ \beta < \alpha: g(\beta) \upharpoonright n = \phi \upharpoonright n \} \subset \{ \beta: \beta < \alpha \} \), denote by \( \phi \downarrow \psi \) such that \( \phi = \theta \uparrow \psi \). Then, for each \( n \in \omega \), \( \{ \beta < \alpha: g(\beta) \upharpoonright 2n = \phi \upharpoonright 2n \} = \{ \beta < \alpha: f(\beta) \upharpoonright n = \theta \upharpoonright n \) and \( f_{\gamma}(\beta) \upharpoonright n = \psi \upharpoonright n \} \subset \{ \beta \in \delta: f(\beta) \upharpoonright n = \theta \upharpoonright n \} \cup \{ \beta \in B: f(\beta) \upharpoonright n = \theta \upharpoonright n \) and \( f_{\gamma}(\beta) \upharpoonright n = \psi \upharpoonright n \}. \) It follows that

\[
\bigcap_{n \in \omega} \{ \beta < \alpha: g(\beta) \upharpoonright n = \phi \upharpoonright n \} \subset \bigcap_{n \in \omega} \{ \beta \in \delta: f(\beta) \upharpoonright n = \theta \upharpoonright n \} \cup \bigcap_{n \in \omega} \{ \beta \in B: f(\beta) \upharpoonright n = \theta \upharpoonright n \} \cap \{ \beta \in \delta: f(\beta) \upharpoonright n = \psi \upharpoonright n \} \]

consequently, as \( \delta, \gamma \in \kappa \sim A \), and \( B \subset \{ \beta \in \gamma: \beta < \gamma, \alpha \} \),

\[
\bigcap_{n \in \omega} \{ \beta < \alpha: g(\beta) \upharpoonright n = \phi \upharpoonright n \} \subset \delta \cup \bigcap_{n \in \omega} \{ \beta < \gamma, \alpha: f_{\gamma}(\beta) \upharpoonright n = \psi \upharpoonright n \} \cap \gamma \]

\[
\subset \delta \cup \{ \beta \in \gamma: \beta < \gamma, \alpha \} = \{ \beta \in \kappa: \beta < \alpha \}. \]

**Lemma 2.5 (V = L).** For a regular \( \kappa \), \( \text{TALS}(\kappa) = \text{TALS}(\kappa) \).

**Proof.** If we use Lemma 2.4 above to substitute for Lemma 1 of [2], the proof of Lemma 2.5 is almost identical to that of Lemma 3 of [2]. We assume that \( \text{TALS}(\kappa) \) holds but there is a totally analytic space \( X \) with underlying set \( \kappa \) such that \( X \) is not \( A \)-left-separated. Then the sets \( A_\alpha \) of Lemma 2.4 are all stationary. It is easily seen that for all \( f, g: \kappa \rightarrow \omega \) and \( \alpha \in \kappa \), we have \( A_\alpha \cap (\alpha + 1) = A_\alpha \cap (\alpha + 1) \) whenever \( f | \alpha = g | \alpha \). Consequently, the sets \( A_\alpha \), for \( f: \kappa \rightarrow \omega \), form what Fleissner calls a stationary system, and we can apply Lemma 2 of [2] to conclude that there exists \( S \subset \kappa \) and \( \Phi(\alpha): \alpha \rightarrow \omega \), for \( \alpha \in S \), such that for each \( f: \kappa \rightarrow \omega \), the set \( \{ \alpha: \Phi(\alpha) = f | \alpha \} \) is a stationary set contained in \( A_\alpha \).

Whenever, \( E \subset \kappa \), \( f \) is a map \( E \rightarrow \omega \) and \( B \subset E \), write

\[
f^{\#}(B) = \bigcup_{\phi \in \omega} \bigcap_{n \in \omega} \{ \beta \in B: f(\beta) \upharpoonright n = \phi \upharpoonright n \}.
\]

The proof of Lemma 3 of [2] works here once we make the following changes: for both \( U_\alpha^f \) and \( U_\alpha \), use \( f^{\#}(\alpha \cap H) \); for \( V_\alpha^f \) and \( V_\alpha \), use \( f^{\#}(\alpha \cap K) \); for \( W_\alpha^f \) and \( W_\alpha \), use \( f^{\#}(\alpha) \); for the notion “\( f^{\#} \) separates \( H \) and \( K \)” use the condition \( f^{\#}(H) \cap f^{\#}(K) = \emptyset \). □

The changes indicated in the above proof also work to show that the results corresponding to Lemmas 4 and 5 of [2] hold in the present situation; in particular, we have the following result:

**Lemma 2.6 (GCH).** For a singular \( \kappa \) of cofinality > \( \omega \), \( \text{TALS}(\kappa) \Rightarrow \text{TALS}(\kappa) \).

The above results complete the proof of Theorem 2.1. Note that since each \( F_\alpha \)-left-separated space is left-separated, we have the following consequence of Theorem 2.1.


**Corollary 2.7 (V = L).** If every subset of a topological space is an $F_\sigma$-set, then the space is left-separated.

Every uncountable left-separated space contains a nonseparable subspace; hence Theorem 2.1 gives the following result.

**Corollary 2.8 (V = L).** Every totally analytic hereditarily separable space is countable.

In light of Theorem 2 of [1], and Theorem 1 of [1], Theorem 4 of [4] or Corollary 4.9 of [6], the following is a consequence of Theorem 2.1.

**Corollary 2.9 (V = L).** If a totally analytic space is either symmetric or semistratifiable, then the space is $F_\sigma$-discrete.

Our second theorem indicates one more particular case, beyond those given in the above corollary, in which the conclusion of Theorem 2.1 can be strengthened to $\sigma$-discreteness. Whereas the proof of Theorem 2.1 was a modification of Fleissner’s proof of the theorem in [2], the proof below consists of an application of that theorem itself. We also use a lemma from [5] on the representation of analytic sets.

**Theorem 2.10 (V = L).** Each totally analytic space of character $\leq \omega_1$ is $\sigma$-discrete.

**Proof.** Let $X$ be a totally analytic space with $\chi(X) \leq \omega_1$. To show that $X$ is $\sigma$-discrete, we construct a normal space $Y$ with $\chi(Y) \leq \omega_1$; $V = L$ and Fleissner’s theorem imply that $Y$ is collectionwise Hausdorff, and from this we infer that $X$ is $\sigma$-discrete.

Let $\mathcal{K}$ be the filter consisting of all sets $H \subset \omega_\omega$ such that there exists $\phi \in \omega_\omega$ such that for each $\psi \in \omega_\omega$, if $\psi \geq \phi$, then there is $n \in \omega$ such that $\{\psi \mid m: m > n\} \subset H$ (these sets are called “final segments of $\omega_\omega$” in [5]). Let $Y = X \times X \times \omega_\omega$. For each $L \subset X$, denote by $L^*$ the subset $\{(x, x, \emptyset) \mid x \in L\}$ of $Y$. For all $H \subset \mathcal{K}$ and $x \in X$, and for each map $g$ from $H$ into the set of all neighborhoods of $x$ in $X$, let

$$U(x, g) = \{(x, x, \emptyset)\} \cup \{\{x\} \times X \times H\} \cup \bigcup_{s \in H} \{g(s) \times \{x\} \times \{s\}\}.$$ 

Topologize $Y$ by making the points of $Y \sim X^*$ isolated and by using the sets $U(x, g)$ to define a neighborhood base for $\{(x, x, \emptyset) \in X^*\}$. Since $|\omega_\omega| = \omega$, $|\mathcal{K}| = 2^\omega$, $\chi(X) \leq \omega_1$, and $2^\omega = \omega_1$ (under $V = L$), it is easily seen that $\chi(Y) \leq \omega_1$. To see that $Y$ is normal, note first that $X^*$ is a closed discrete subset of $Y$ with discrete complement. Consequently, $Y$ is normal provided that for each $L \subset X$, the sets $L^*$ and $(X \sim L)^*$ can be separated by open subsets of $Y$. Let $L \subset X$. Since $X$ is totally analytic, there are closed subsets $F_s$ and $K_s$, $s \in \omega_\omega$, of $X$ such that $L = \bigcup_{\phi \in \omega_\omega} \bigcap_{n \in \omega} F_{\phi n}$ and $X \sim L = \bigcup_{\phi \in \omega_\omega} \bigcap_{n \in \omega} K_{\phi n}$; by Lemma 2 of [5], we can choose the sets $F_s$ and $K_s$ so that for all $\phi, \psi \in \omega_\omega$ and $n \in \omega$, $F_{\phi n+1} \subset F_{\phi n}$ and $K_{\phi n+1} \subset K_{\psi n}$, and, if $\phi \leq \psi$, then $F_{\phi n} \subset F_{\psi n}$ and $K_{\phi n} \subset K_{\psi n}$. For each $x \in X$, let $H_x = \{s \in \omega_\omega : x \in F_s \sim K_s\}$ and $J_x = \{s \in \omega_\omega : x \in K_s \sim F_s\}$; note that if $x \in L$, then $H_x \in \mathcal{K}$ and if $x \in X \sim L$, then $J_x \in \mathcal{K}$. For each $s \in \omega_\omega$, let $g(s) = X \sim K_s$ and $f(s) = X \sim F_s$. Let $V = \bigcup_{s \in \omega_1} U(x, g \mid H_x)$ and $W = \bigcup_{x \in X \sim L} U(x, f \mid J_x)$, and note that these are open subsets of $Y$ containing $L^*$ and...
We show that $V \cap W = \emptyset$. Assume on the contrary that there are $x \in L$, $z \in X \sim L$ and $(u, v, s) \in U(x, g | H_x) \cap U(z, f | J_x)$. We have $s \in H_x \cap J_x$ and it follows that $x \in F_s$ and $z \in K_s$. Note that either $u = x$ or $u = z$. If $u = x$, then $x \in f(s) = X \sim F_s$, and if $u = z$, then $z \in g(s) = X \sim K_s$; hence we have a contradiction in both cases.

By Fleissner's theorem in [2], $Y$ is collectionwise Hausdorff. Consequently, there are $H_x \in \mathfrak{R}$, $x \in X$, and for each $x \in X$, a map $g_x$ from $H_x$ to the neighborhoods of $x$ in $X$ such that for any two distinct points $x$ and $y$ of $X$, $U(x, g_x) \cap U(y, g_y) = \emptyset$.

For each $s \in \omega_1$, let $D_s = \{x \in X : s \in H_x \}$. Then $X = \bigcup_{s \in \omega_1} D_s$, and for each $s \in S$, the subspace $D_s$ of $X$ is discrete because, for each $x \in D_s$ and for any $y \in D_s \sim \{x\}$, we have $(y, x, s) \in U(y, g_x) \sim U(x, g_x)$ and hence $y$ cannot belong to the neighborhood $g_x(s)$ of $x$. □

**Remarks.**

1°. Note that the construction used above can be used (with a slight change in the proof of normality of $Y$) to show that, under $V = L$, a space of character $\leq \omega_1$ is $\sigma$-discrete provided that each subset of the space is $\mathfrak{V}$-analytic (i.e., can be obtained by the $\mathcal{A}$-operation from a family of open sets).

2°. In a $\sigma$-discrete space, every subset is a Borel-set (more precisely, the countable union of sets, each of which is the intersection of an open set with a closed set). In spite of Theorem 2.10, the following problem remains open: Is it true under $V = L$ that a space of character $\leq \omega_1$ has to be $\sigma$-discrete if each subset of the space is a Borel-set? The one-point compactification of an uncountable discrete space provides an example of a space in which each subset is Borel while there are nonanalytic subsets (note that if $x \in X$ is such that $X \sim \{x\}$ is analytic, then $\{x\}$ is a $G_\delta$-set).

3. Some related results. An interesting problem motivated by the previous results is whether a metrizable space has to be $\sigma$-discrete under $V = L$ provided each subset of the space is projective. We have been unable to solve that problem even for the first level of projectivity ($\Sigma^1_1$) beyond analyticity. A related problem is whether a metrizable space has to be $\sigma$-discrete under $V = L$ in case each subset of the space belongs to the $\sigma$-algebra generated by the analytic subsets of the space. A positive solution to the latter problem is contained in the following results.

Let us call a subset $A$ of a topological space $X$ strongly analytic if there are open subsets $G_s$ of $X$, for $s \in \omega_1$, such that

$$A = \bigcup_{\phi \in \omega_1} \bigcap_{n \in \omega} G_{\phi|n} = \bigcup_{\phi \in \omega_1} \bigcap_{n \in \omega} \overline{G_{\phi|n}}.$$ 

Each strongly analytic set is both analytic and $\mathfrak{S}$-analytic. On the other hand, it is easy to see that in a perfectly normal space, every analytic set is strongly analytic.

**Proposition 3.1** $(V = L)$. If $X$ is a space of character $\leq \omega_1$ such that each subset of $X$ belongs to the $\sigma$-algebra generated by the strongly analytic subsets of $X$, then $X$ is $\sigma$-discrete.

**Proof.** Assume that $X$ satisfies the conditions stated in the proposition. Denote by $\mathfrak{S}(X)$ (by $\mathfrak{S}$) the family consisting of all (strongly analytic) subsets of $X$. For any family $\mathfrak{C}$ of subsets of $X$, denote by $\mathfrak{C}_\sigma$ (by $\mathfrak{C}_\delta$) the family consisting of all countable...
unions (intersections) of sets from $\mathcal{C}$, and let $\lnot S = \{X \sim L: L \in \mathcal{L}\}$. Let $S_0 = S \cup \lnot S$, and define $S_\alpha$ for $\alpha < \omega_1$, recursively by the formula $S_\alpha = \mathcal{C}_\alpha \cup \lnot S_\alpha$, where $\mathcal{C}_\alpha = (\bigcup_{\beta < \alpha} S_\beta) \cup (\bigcup_{\beta < \alpha} \lnot S_\beta)$. Then $\bigcup_{\alpha < \omega_1} S_\alpha$ is the $\sigma$-algebra generated by $S$; consequently, $\bigcup_{\alpha < \omega_1} S_\alpha = \mathcal{P}(X)$. It follows by $V = L$ (or just CH) and Theorem 9 of [3] that there exists $\rho < \omega_1$ such that $S_\rho = \mathcal{P}(X)$.

The remainder of this proof is carried out in two stages. First we use the topology on $X \times \langle (\omega) \cup \omega' \rangle$ defined by Hansell in the proof of the theorem in [5] to obtain a space $Y$ such that for each $S \in \mathcal{S}$, the sets $S \times \langle \omega \rangle$ and $(X \sim S) \times \langle \omega \rangle$ can be separated by open subsets of $Y$. Then we define a set $I \subset \rho + 1$ and a topology on $Y \times \langle (\omega) \cup I \rangle$ to obtain a space $Z$ such that for each $S \in \mathcal{S}_\rho$, the sets $S \times \langle (\omega) \cup \omega' \rangle$ and $(X \sim S) \times \langle (\omega) \cup \omega' \rangle$ can be separated by open subsets of $Z$; since $\mathcal{S}_\rho = \mathcal{P}(X)$, the space $Z$ turns out to be normal and, by Fleissner's theorem, collectionwise Hausdorff; finally, collectionwise Hausdorffness of $Z$ implies that $X$ is $\sigma$-discrete.

Let $Y = X \times \langle (\omega) \cup \omega' \rangle$ and topologize $Y$ as follows. Make all the points of $X \sim X \times \langle \omega \rangle$ isolated. Let the filter $\mathcal{K}$ on $\omega'$ be defined as in the proof of Theorem 2.10. A neighborhood-base for a point $\langle x, \omega \rangle \in X \times \langle \omega \rangle$ consists of all sets of the form

$$U(x, g) = \{\langle x, \omega \rangle\} \cup \bigcup_{s \in H} (g(s) \times \{s\}),$$

where $H \in \mathcal{K}$ and $g$ is a map with domain $H$ and range a set of neighborhoods of $x$ in $X$. As $\chi(X) \leq \omega_1$, it is easily seen that $\chi(Y) \leq \omega_1$. Let $S$ be a strongly analytic subset of $X$. We show that $S \times \langle (\omega) \cup \omega' \rangle$ can be separated by open subsets of $Y$. Let $G_\phi, s \in \omega'$, be open subsets of $X$ such that $S = \bigcup_{\phi \in \omega'} \bigcap_{n \in \omega} G_{\phi n} = \bigcup_{s \in \omega'} \bigcap_{n \in \omega} G_{\phi n}$. The proof of Lemma 2 of [5] shows that we can choose the sets $G_\phi$ so that for all $\phi, \psi \in \omega'$ and $n \in \omega$, $G_{\phi n + 1} \subseteq G_{\phi n}$ and, if $\phi \leq \psi$, then $G_{\phi n} \subseteq G_{\psi n}$. For each $x \in X$, let $H_\phi = \{s \in \omega': x \in G_\phi \}$ and $J_\phi = \{s \in \omega': x \notin G_\phi \}$; note that if $x \in S$, then $H_\phi \in \mathcal{K}$ and if $x \in X \sim S$, then $J_\phi \in \mathcal{K}$. For each $s \in \omega'$, let $g(s) = G_\phi$ and $f(s) = X \sim G_\phi$. Let $V = \bigcup_{x \in X} U(x, g \mid H_x)$ and $W = \bigcup_{x \in X} U(x, f \mid J_x)$. Then $V$ and $W$ are open subsets of $Y$ containing $S \times \langle (\omega) \cup \omega' \rangle$, respectively. It is easy to see that $V \cap W = \emptyset$.

Let $I$ be the set of all step-functions in $\rho + 1$ (i.e., $f \in I$ iff $f \in \rho + 1$ and there are ordinals $0 = \alpha_0 < \cdots < \alpha_n = \rho + 1$, for some $n \in \omega$, such that $f$ is constant on each of the intervals $[\alpha_i, \alpha_{i+1}] \subseteq \rho + 1$). Note that $|I| = \omega$. We use induction on $\alpha$ to define a filter-base $\mathcal{B}_\alpha$ on $I$. Let

$$\mathcal{B}_0 = \{\{f \in I: f(0) \neq n\}: n < \omega\}.$$

If $\mathcal{B}_\beta$ has been defined for each $\beta < \alpha \leq \rho$, then $\mathcal{B}_\alpha$ consists of all sets of the form $\bigcup_{k \geq n} \{s \in E_k: s(\alpha) = k\}$, where $n \in \omega$ and $E_k \subseteq \bigcup_{\beta < \alpha} \mathcal{B}_\beta$ for each $k \geq n$. Note that for all $\beta < \alpha \leq \rho$, $\mathcal{B}_\beta \subseteq \mathcal{B}_\alpha$; consequently, $\bigcup_{\alpha < \rho} \mathcal{B}_\alpha = \mathcal{B}_\rho$. Each of the families $\mathcal{B}_\alpha$ is closed under finite intersections: this holds trivially for $\alpha = 0$; if it holds for each $\alpha < \gamma$, and if $n, m \in \omega$ and $E_k, H_i \in \bigcup_{\alpha < \gamma} \mathcal{B}_\alpha$ for $k \geq n$ and $i \geq m$, then

$$\bigcup_{k \geq n} \{s \in E_k: s(\gamma) = k\} \cap \bigcup_{i \geq m} \{s \in H_i: s(\gamma) = i\} = \bigcup_{j \geq \max(n, m)} \{s \in E_j \cap H_j: s(\gamma) = j\} \in \mathcal{B}_\gamma.$$
so that $\mathcal{E}_\gamma$ is closed under finite intersections. In particular, $\mathcal{E}_\rho$ is closed under finite intersections. Each $\mathcal{E}_\alpha$ consists of nonempty sets. To see this, we use induction on $\alpha$ to show that for each $\alpha \leq \rho$, and for all $f \in I$ and $E \in \mathcal{E}_\alpha$, there exists $g \in E$ such that $g|\{\beta: \alpha < \beta \leq \rho\} = f|\{\beta: \alpha < \beta \leq \rho\}$. The statement holds trivially for $\alpha = 0$; assume it holds for each $\alpha < \gamma$. Let $f \in I$ and $E \in \mathcal{E}_\gamma$. Let $n \in \omega$ and $E_k = \bigcup_{\alpha < \gamma} \mathcal{E}_\alpha$, for $k \geq n$, be such that $E = \bigcup_{k \geq n} (s \in E_k : s(\gamma) = k)$. Define $f' \in \omega^+ \omega$ by setting $f'(\gamma) = n$ and $f'(\alpha) = f(\alpha)$ for each $\alpha \neq \gamma$; note that $f' \in I$. Let $\delta < \gamma$ be such that $E_n \in \mathcal{E}_\delta$. By the induction assumption, there exists $g \in E_n$ such that

$$g|\{\beta: \delta < \beta \leq \rho\} = f'|\{\beta: \delta < \beta \leq \rho\}.$$  

Clearly, $g \in E$ and $g|\{\beta: \gamma < \beta \leq \rho\} = f|\{\beta: \gamma < \beta \leq \rho\}$; this completes the induction. It follows that $\mathcal{E}_\rho$ consists of nonempty sets; hence $\mathcal{E}_\rho$ is a filter-base.

Let $Z = Y \times (\{\omega\} \cup I)$. Topologize $Z$ as follows. Make all the points of $Z \sim X \times \{\omega\} \times \{\omega\}$ isolated. A neighborhood-base for a point $\langle x, \omega, \omega \rangle \in X \times \{\omega\} \times \{\omega\}$ consists of all sets of the form

$$V(x, g) = \{\langle x, \omega, \omega \rangle\} \cup \bigcup_{f \in E} \{\varphi(f) \times \{f\}\},$$

where $E \in \mathcal{E}_\rho$ and $\varphi$ is a map with domain $E$ and range a set of neighborhoods of $\langle x, \omega \rangle$ in $Y$. Using that $\chi(Y) \leq \omega_1$, we easily see that $\chi(Z) \leq \omega_1$. Note that $X \times \{\omega\} \times \{\omega\}$ is a closed and discrete subset of $Z$ with discrete complement; hence, to show that $Z$ is normal, it suffices to show that for each $S \in \mathcal{E}(X)$, the sets $S \times \{\omega\} \times \{\omega\}$ and $(X \sim S) \times \{\omega\} \times \{\omega\}$ can be separated by open subsets of $Z$. Call an open subset $G$ of $Z$ $\alpha$-open, where $\alpha \leq \rho$, provided that for each $\langle x, \omega, \omega \rangle \in G \cap X \times \{\omega\} \times \{\omega\}$, there exists a neighborhood $V(x, \varphi)$ of $\langle x, \omega, \omega \rangle$ contained in $G$ such that $\text{dom}(\varphi) \in \mathcal{E}_\alpha$. For each $\alpha \leq \rho$, denote by $\mathcal{E}_\alpha$ the family consisting of all sets $R \subset X$ such that the sets $R \times \{\omega\} \times \{\omega\}$ and $(X \sim R) \times \{\omega\} \times \{\omega\}$ can be separated by $\alpha$-open subsets of $Z$; note that $\mathcal{E}_\alpha$ is closed under complementation, and because of the corresponding property of $\mathcal{E}_\alpha$, $\mathcal{E}_\alpha$ is closed under finite intersections. We use induction on $\alpha$ to show that, for each $\alpha \leq \rho$, $\mathcal{E}_\alpha \subset \mathcal{E}_\alpha$. For each $S \in \mathcal{E}$, the set $S \times \{\omega\}$ and $(X \sim S) \times \{\omega\}$ can be separated by open sets in $Y$; from this it follows that $\mathcal{E}_0 \subset \mathcal{E}_0$. Let $\alpha$ be such that $0 < \alpha \leq \rho$ and, for each $\beta < \alpha$, $\mathcal{E}_\beta \subset \mathcal{E}_\alpha$. To show that $\mathcal{E}_\alpha \subset \mathcal{E}_\alpha$, recall that $\mathcal{E}_\alpha = \mathcal{E}_\alpha \cup \mathcal{E}_\alpha$, where $\mathcal{E}_\alpha = (\bigcup_{\beta < \alpha} \mathcal{E}_\beta) \cup (\bigcup_{\beta > \alpha} \mathcal{E}_\beta)$; since $\mathcal{E}_\beta$ is an algebra of sets for each $\beta \leq \alpha$, the inclusion $\mathcal{E}_\alpha \subset \mathcal{E}_\alpha$ follows once we show that for any decreasing sequence $\langle S_n \rangle_{n < \omega}$ of sets from $\bigcup_{\beta < \alpha} \mathcal{E}_\beta$, the set $\bigcap_{n < \omega} S_n$ belongs to $\mathcal{E}_\alpha$. Let $S_n, n < \omega$, be members of $\bigcup_{\beta < \alpha} \mathcal{E}_\beta$ such that for each $n < \omega$, $S_{n+1} \subset S_n$. For each $n < \omega$, $S_n \in \bigcup_{\beta < \alpha} \mathcal{E}_\beta$ and hence there are maps $\varphi_{x,n}$, with domain a member of $\bigcup_{\beta < \alpha} \mathcal{E}_\beta$ and range a set of neighborhoods of $\langle x, \omega \rangle$ in $Y$, for each $x \in X$, such that for all $x \in S_n$ and $y \in X \sim S_n$, $V(x, \varphi_{x,n}) \cap V(y, \varphi_{x,n}) = \emptyset$. Define the maps $\varphi_x$, $x \in X$, as follows. Let $x \in X$. If $x \in \bigcap_{n < \omega} S_n$, set $n(x) = 0$ and if $x \in X \sim \bigcap_{n < \omega} S_n$, let $n(x) < \omega$ be such that $x \notin S_{n(x)}$. Let $E = \bigcup_{k > n(x)} \{f \in \text{dom}(\varphi_{x,k}) : f(\alpha) = k\}$, and note that $E_x \in \mathcal{E}_\alpha$. We make $E_x$ the domain of $\varphi_x$, and for each $f \in E_x$, we let $\varphi_x(f)$ be the neighborhood $\varphi_x(f)(\alpha)$ of $\langle x, \omega \rangle$ in $Y$. Clearly, $\bigcup \{V(x, \varphi_x) : x \in \bigcap_{n < \omega} S_n\}$ and
\[ \{ V(x, \varphi_x) : x \in X \sim \cap_{n<\omega} S_n \} \] are \( \alpha \)-open subsets of \( Z \) containing \(( \cap_{n<\omega} S_n ) \times \{ \omega \} \times \{ \omega \} \) and \(( X \sim \cap_{n<\omega} S_n ) \times \{ \omega \} \times \{ \omega \} \), respectively. To see that these sets are disjoint, assume on the contrary that there are \( x \in \cap_{n<\omega} S_n \) and \( y \in X \sim \cap_{n<\omega} S_n \) such that \( V(x, \varphi_x) \cap V(y, \varphi_y) \neq \emptyset \). Then there exists \( f \in \text{dom}(\varphi_x) \cap \text{dom}(\varphi_y) \) such that \( \varphi_x(f) \cap \varphi_y(f) \neq \emptyset \). Let \( k = f(\alpha) \). Since \( f \in \text{dom}(\varphi_x) \), we have \( k \geq n(y) \) and hence \( y \in X \sim S_k \). It follows that \( V(x, \varphi_{x,k}) \cap V(y, \varphi_{y,k}) = \emptyset \) and further, since \( f \in \text{dom}(\varphi_{x,k}) \cap \text{dom}(\varphi_{y,k}) \), that \( \varphi_{x,k}(f) \cap \varphi_{y,k}(f) = \emptyset \); however, this is a contradiction, since \( \varphi_{x,k}(f) = \varphi_x(f) \) and \( \varphi_{y,k}(f) = \varphi_y(f) \). We have shown that \( \cap_{n<\omega} S_n \subseteq \cap_{\alpha} \). This completes the proof of the inductive step. Since \( \cap_{\alpha} \subseteq \cap_{1} \) and \( \cap_{1} = \cap_z \), we have proved that \( Z \) is normal.

By the theorem in [2], \( Z \) is collectionwise Hausdorff. Consequently, we can find mutually disjoint basic neighborhoods \( V(x, \varphi_x) \) for the points \( (x, \omega, \omega) \) of the closed discrete set \( X \times \{ \omega \} \times \{ \omega \} \). For each \( x \in X \), let \( f_x \in \text{dom}(\varphi_x) \), and choose a basic neighborhood \( U(x, g_x) \) contained in the neighborhood \( \varphi_x(f_x) \) of \( (x, \omega) \) in \( Y \), and let \( s_x \in \text{dom}(g_x) \); note that \( g_x(s_x) \) is a neighborhood of \( x \) in \( X \). For all \( f \in I \) and \( s \in \omega^\omega \), let \( D_{f,s} = \{ x \in X : f_x = f \text{ and } s_x = s \} \); note that \( D_{f,s} \) is a discrete subspace of \( X \) since for all distinct \( x \) and \( y \) in \( D_{f,s} \), \( g_x(s_x) \cap g_y(s_y) = \emptyset \). Since \( | I \times \omega^\omega | = \omega \), it follows that \( X \) is \( \sigma \)-discrete.

**Corollary 3.2** \(( V = L ) \). If \( X \) is a perfectly normal space of character \( \leq \omega_1 \) such that each subset of \( X \) belongs to the \( \sigma \)-algebra generated by the analytic subsets of \( X \), then \( X \) is \( F_\sigma \)-discrete.

Since each metrizable space is perfectly normal and has character \( \leq \omega_1 \), Corollary 3.2 and Proposition 2.2 of [7], give the following result.

**Corollary 3.3** \(( V = L ) \). If \( X \) is a \( \alpha \)-space such that each subset of \( X \) belongs to the \( \sigma \)-algebra generated by the analytic subsets of \( X \), then \( X \) is \( F_\sigma \)-discrete.

**References**


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