EMBEDDINGS IN MINIMAL HAUSDORFF SPACES

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Abstract. We show that not every semiregular space is embeddable as an open and dense set of some minimal Hausdorff space. Also a space is constructed for which it is not decidable in Z.F.C. whether such an embedding exists.

1. Introduction. In this paper we investigate the following question.

Question. Is a semiregular space \( X \) embeddable in a minimal Hausdorff space as a dense and open subset?

(Note that a Hausdorff space is called minimal Hausdorff if it contains no strictly coarser Hausdorff topology and that a space \( Y \) is called semiregular if \( \{ \text{int} (\text{cl} A) : A \subseteq Y \} \) is an open basis for \( Y \).)

This question appeared in the paper [Ve] and was motivated by the following embedding theorem.

**Theorem A.** Let \( X \) be a semiregular space. Then:

(i) [Ka] \( X \) is embeddable as a dense subset of a minimal Hausdorff space.

(ii) [Ve] \( X \) is embeddable as an open subset of a minimal Hausdorff space.

(iii) [Ve] The space \( X \oplus X \) — two disjoint copies of \( X \) — is embeddable as a dense and open subspace of a minimal Hausdorff space. □

We present two examples which show the following.

**Example 1.** There exists a zero-dimensional Lindelöf space \( X \) for which the answer to the question is negative.

**Example 2.** There exists a zero-dimensional Lindelöf space \( X \) for which the question cannot be answered in Z.F.C. without additional assumptions.

For these examples we use the following notions.

A function \( f : X \to Y \) is called irreducible, whenever \( f \) is surjective and \( f(A) \neq Y \), for every closed subset \( A \subseteq X \).

A function \( f : X \to Y \) is called \( \theta \)-continuous if for each \( x \in X \) and each neighborhood \( U \) of \( f(x) \), there is a neighborhood \( V \) of \( x \) such that \( f(\text{cl} V) \subseteq \text{cl} U \).

The absolute of a space \( X \) is the unique semiregular and extremally disconnected space \( EX \) which can be mapped onto \( X \) by a perfect, irreducible and \( \theta \)-continuous function \( \pi \).
If $\alpha D$ is a minimal Hausdorff extension of a discrete space $D$, then $E\alpha D = \beta D$ and the map $\pi: E\alpha D \to \alpha D$ is the (unique) $\theta$-continuous extension of $\text{id}: D \to \alpha D$ to $\beta D$.

The next theorem is the key to the examples.

**Theorem B** [VeW]. Let $X$ be a compact space. Let $f: X \to Y$ be a compact and irreducible function onto a set $Y$. Then the collection \{ $f(B)$: $B$ is a closed subset of $X$ \} is a closed base for a topology $\theta(f)$ on $Y$, the space $(Y, \theta(f))$ is minimal Hausdorff and the function $f: X \to (Y, \theta(f))$ is $\theta$-continuous.

**Remark.** The definitions of all undefined notions we used in the previous theorem can be found in the excellent survey paper [Wo] of R. G. Woods.

2. **The example.** Let $D$ be a discrete space with $\text{card } D \geq 2^{2^{\omega_0}}$ and let $X = D \cup \{\omega\}$ be the one-point Lindelöfication of $D$. In particular, the collection \{ $D' \cup \{\omega\}$: $D'$ a cocountable subset of $D$ \} is a local base at $\omega$ in $X$.

**Theorem C.** If $\text{card } X > 2^{2^{\omega_0}}$, then $X$ is not embeddable as a dense and open subset in a minimal Hausdorff space.

**Proof.** Assume the opposite, say $X$ is embedded in the minimal Hausdorff space $Y$ as a dense and open subset.

Then the space $Y$ can be considered as a minimal Hausdorff extension of the discrete space $D$, say $Y = \alpha D$ and $D \subset X \subset \alpha D$.

Then $E\alpha D = \beta D$ and the absolute function $\pi: \beta D \to \alpha D$ is the unique $\theta$-continuous extension of $\text{id}: D \to \alpha D$ to $\beta D$.

We observe the following facts:

(i) Since the function $\pi: \beta D \to \alpha D$ is perfect and irreducible and the space $\alpha D$ is minimal Hausdorff, the collection \{ $\pi(B)$: $B$ is a closed subset of $\beta D$ \} is a closed base for the topology on $\alpha D$.

(ii) $\pi^{-1}\{\omega\} = \{ \mathcal{F}: \mathcal{F} \text{ an ultrafilter on } D \text{ with: } \forall F \in \mathcal{F}: \text{card } F > \omega_0 \} \subset \beta D - D$.

(iii) If $B$ is a compact subset of $\beta D$ with $B \cap \pi^{-1}\omega = \emptyset$, then it is easy to see that $\text{card } B \leq 2^{2^{\omega_0}}$.

In particular, $\text{card } \pi^{-1}\{d\} \leq 2^{2^{\omega_0}}$ for each $d \in \alpha D - X$.

(iv) Since $\text{card } D > 2^{2^{\omega_0}}$, $\text{card}(\beta D - \pi^{-1}\{\omega\}) > 2^{2^{\omega_0}}$. Since $X$ is an open subset of $\alpha D$, then by (i), there exists a compact subset $B \subset \beta D$ such that $\alpha D - X \subset \pi(B) \subset \alpha D - \{\omega\}$. However, by (iii), $B \leq 2^{2^{\omega_0}}$ and $\text{card } \pi^{-1}\{p\} \leq 2^{2^{\omega_0}}$ for each $p \in \alpha D - \{\omega\}$.

Consequently, $\text{card } \pi^{-1}(\pi(B)) \leq 2^{2^{\omega_0}}$. We conclude, from (iv), that $\beta D - \pi^{-1}\{\omega\}$ is not a subset of $\pi^{-1}(\pi(B))$, which contradicts that $\alpha D - X \subset \pi(B)$. This completes the proof of the theorem.

**Theorem D.** If $\text{card } X = 2^{2^{\omega_0}}$, the space $X$ is embeddable as a dense and open subset of a minimal Hausdorff space.

**Proof.** We construct such an extension of $X$ as follows. Consider the space $\beta D$ and define the closed subset $A \subset \beta D - D$ by

$$A = \{ \mathcal{F}: \mathcal{F} \text{ is an ultrafilter on } D \text{ and } \text{card } F > \omega_0 \text{ for each } f \in \mathcal{F} \}.$$
Note that $\beta D - A = \bigcup \{ \text{cl } D' : D' \text{ is a countable subset of } D \}$. We conclude that $\text{card}(\beta D - D - A) = (2^{2^{\aleph_0}})^{\omega_0} \cdot 2^{2^{\aleph_0}} = 2^{2^{\aleph_0}}$. Fix a countable set $\mathcal{N} \subseteq D$. Then $\text{cl } \mathcal{N} \cap A = \emptyset$ and $\text{card}(\text{cl } \mathcal{N} - \mathcal{N}) = 2^{2^{\aleph_0}}$. Let $g: \text{cl } \mathcal{N} - N \rightarrow \beta D - D - A - \text{cl } \mathcal{N}$ be a bijection between these sets. Define a partition $E$ of $\beta D$ by

$$E = \{ \{ d \} : d \in D \} \cup \{ A \} \cup \{ \{ x, g(x) \} : x \in \text{cl } \mathcal{N} - \mathcal{N} \}.$$  

Let $Y$ denote the set $\beta D \mod E$. The corresponding quotient function $f: \beta D \rightarrow Y$ is a compact and irreducible surjection.

Consider the minimal Hausdorff topology $\theta(f)$ on $Y$, as defined in Theorem B. The following properties are easy to verify:

(i) $f(D)$ is an open discrete and dense subset of $Y$,
(ii) the subspace $f(D) \cup \{ A \}$ of $Y$ is homeomorphic to the space $X$,
(iii) $f(D) \cup \{ A \}$ is dense in $Y$ (since $f(D) \subseteq f(D \cup A)$), and
(iv) $f(A) \cup \{ A \}$ is open in $Y$ (since $Y - f(D \cup A) = f(\text{cl } \mathcal{N} - \mathcal{N})$).

These properties show that we have embedded the space $X$ as a dense and open subset of the minimal Hausdorff space $Y$.

**Remarks.** There are many cardinals $\chi$, e.g. $\chi = 2^{2^{\omega}}$, $\chi = \aleph_3$, for which it is consistent to assume that $\chi > 2^{2^{\omega}}$ and consistent to assume that $\chi = 2^{2^{\omega}}$. Thus, if $D$ is a discrete space of cardinality $\chi$, then $X$ can be embeddable, by Theorem D, or not embeddable, by Theorem C, as a dense and open subspace of some minimal Hausdorff space depending on set-theoretic assumptions of whether $\chi = 2^{2^{\omega}}$ or $\chi > 2^{2^{\omega}}$.

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**References**


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