ADJUNCTION SPACES OF MONOTONICALLY NORMAL SPACES
AND SPACES DOMINATED BY
MONOTONICALLY NORMAL SUBSETS

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ABSTRACT. In this paper, we shall prove the following results: (1) the
adjunction space of two monotonically normal spaces is also monotonically
normal, (2) a topological space is monotonically normal if and only if it is
dominated by a collection of monotonically normal subsets.

The class of monotonically normal spaces was introduced by P. Zenor, and
studied by P. Zenor, R. Heath and D. Lutzer [5], C. R. Borges [2, 3] and others.
In this paper, we shall first prove that the adjunction space of two monotonically
normal spaces is monotonically normal, and as an immediate corollary that a
topological space is an AR(M.N) (resp. ANR(M.N)) if and only if it is an AE(M.N)
(resp. ANE(M.N)), where M.N is the class of all monotonically normal spaces.

In [5, p. 483], it was shown that if a topological space X can be covered by a
locally finite (or even hereditarily closure preserving) collection of closed monotonically
normal subspaces, then X is monotonically normal. Concerning this result, it
was asked whether a topological space X must be monotonically normal provided
that X is dominated by a collection such subspaces. In §2, we shall answer this
question affirmatively.

Throughout this paper, all spaces are assumed to be Hausdorff topological spaces.
Clx denotes the closure operator in a space X.

1. Adjunction spaces. In this paper, as the definition of monotonically normal
space, we exclusively use [5, Lemma 2.2(a)]; i.e. a space X is monotonically normal
if there is a function G which assigns to each pair (A, B) of separated subsets of X
an open set G(A, B) satisfying

(i) \( A \subset G(A, B) \subset Cl_x G(A, B) \subset X - B. \)
(ii) if \((A', B')\) is a pair of separated subsets having \(A \supset A'\) and \(B \supset B'\), then
\[ G(A, B) \subset G(A', B'), \]
where two subsets A, B of X are separated if \(A \cap Cl_x B = \emptyset, Cl_x A \cap B = \emptyset.\)

The function G is called a monotone normality operator for X. We put \(G(\emptyset, B) = \emptyset, G(A, \emptyset) = X,\) where \(A \neq \emptyset.\) These are used in the proof of Theorem 1.1; for instance, \(G_2(A_Y, B_Y, G_1(A_1, B_1)\) if \(A_Y = \emptyset, B_Y = \emptyset, A_1 = \emptyset\) or \(B_1 = \emptyset.\)

THEOREM 1.1. Let \(X\) and \(Y\) be monotonically normal spaces, \(F\) a closed subset
of \(X\) and \(f: F \rightarrow Y\) a continuous mapping. Then the adjunction space \(X \cup_f Y = Z\)
is monotonically normal.
ADJUNCTION SPACES OF MONOTONICALLY NORMAL SPACES

PROOF. Let $h: X \to Z$, $k: Y \to Z$ be the natural projections. By definition of $X \cup Y$, $U \subset Z$ is open (closed) if and only if $h^{-1}(U)$ and $k^{-1}(U)$ are open (closed); furthermore, $k$ and $h|X - F$ are homeomorphisms into. For convenience, for a subset $A$ of $Z$, we let $A_X = h^{-1}(A)$, $A_Y = k^{-1}(A)$.

Let $G_1$ and $G_2$ be monotone normality operators for $X$ and $Y$, respectively, and $(A, B)$ a pair of separated subsets of $Z$. We shall show that there is a monotone normality operator $G$ for $Z$ such that $k^{-1}(G(A, B)) = G_2(A_Y, B_Y)$.

For a pair $(A, B)$ of separated subsets of $Z$, $(A_Y, B_Y)$ is a pair of separated subsets of $Y$. Let

$$A_1 = A_X \cup f^{-1}(G_2(A_Y, B_Y)),$$
$$B_1 = (F - f^{-1}(C_Y G_2(A_Y, B_Y))) \cup B_X.$$

Then $(A_1, B_1)$ is a pair of separated subsets of $X$. Since there is an open subset $G_1(A_1, B_1)$ of $X$ such that

$$A_1 \subset G_1(A_1, B_1) \subset Cl_X G_1(A_1, B_1) \subset X - B_1,$$

there is an open subset $U_A$ of $X$ such that

$$U_A - F = G_1(A_1, B_1) - F, \quad U_A \cap F = f^{-1}(G_2(A_Y, B_Y)).$$

Let $G(A, B) = h(U_A) \cup k(G_2(A_Y, B_Y))$. Then it is easily seen that $G$ is a monotone normality operator for $Z$ such that $k^{-1}(G(A, B)) = G_2(A_Y, B_Y)$. This completes the proof.

AR(C) (resp. ANR(C)) is the abbreviation for absolute (resp. neighborhood) retract for the class $C$ and AE(C) (resp. ANE(C)) the abbreviation for absolute (resp. neighborhood) extensor for the class $C$. For these definitions, see [6]. Note that, in [4], ES(C) and NES(C) are used instead of AE(C) and ANE(C), respectively.

COROLLARY 1.2. Let $X$ be a monotonically normal space. Then $X$ is an AR(MAN) (resp. ANR(MAN)) if and only if $X$ is an AE(MAN) (resp. ANE(MAN)).

PROOF. This follows from Theorem 1.1, using the same method of proof of Theorem 8.1 in [4].

2. Spaces dominated by monotonically normal subsets. We start by reproducing Definition 8.1 in [7].

DEFINITION 2.1. Let $X$ be a space, and $B$ a collection of closed subsets of $X$. Then $B$ dominates $X$ if, whenever $A \subset X$ has a closed intersection with every element of some subcollection $B_1$ of $B$ which covers $A$, then $A$ is closed.

In [7] (resp. [1]), it is shown that a space is paracompact (resp. stratifiable) if and only if it is dominated by a collection of paracompact (resp. stratifiable) subspaces. We prove the following:

THEOREM 2.2. A space is monotonically normal if and only if it is dominated by a collection of monotonically normal subsets.

PROOF. Since the "only if" part is trivial, we prove the "if" part. Let $B$ be a dominating collection of monotonically normal subsets of a space $X$. Consider the class $G$ of all pairs of the form $(C_\alpha, G_\alpha)$, where $C_\alpha \subset B$, and $G_\alpha$ is a monotone normality operator for $C_\alpha = \bigcup C_\alpha$. (Throughout this proof, $\bigcup C_\gamma$ will be denoted by $C_\gamma$ for any $C_\gamma \subset B$.) We partially order $G$ by letting $(C_\alpha, G_\alpha) \leq (C_\beta, G_\beta)$ whenever $C_\alpha \subset C_\beta$ and, for each pair $(A, B)$ of separated subsets of $C_\beta$, $G_\beta(A, B) \cap C_\alpha = G_\alpha(A \cap C_\alpha, B \cap C_\alpha)$. 


We now show that any linearly ordered subfamily \( \{(C_\alpha, G_\alpha) : \alpha \in \Lambda \} \) of \( \mathcal{G} \) has an upper bound \((C_\beta, G_\beta)\). Let \( \mathcal{C}_\beta = \bigcup \{C_\alpha : \alpha \in \Lambda \} \). For each pair \((A, B)\) of separated subsets of \( C_\beta \), let

\[
G_\beta(A, B) = \bigcup \{G_\alpha(A \cap C_\alpha, B \cap C_\alpha) : \alpha \in \Lambda \},
\]

and let us show that \( G_\beta \) is a monotone normality operator for \( C_\beta \). In fact, first, since \( G_\beta(A, B) \cap C_\alpha = G_\alpha(A \cap C_\alpha, B \cap C_\alpha) \) for each \( \alpha \in \Lambda \), \( G_\beta(A, B) \) is open in \( C_\beta \). Next, let

\[
K_\beta = \bigcup \{\text{Cl}_{C_\alpha} G_\alpha(A \cap C_\alpha, B \cap C_\alpha) : \alpha \in \Lambda \}.
\]

Clearly \( G_\beta(A, B) \subseteq K_\beta \subseteq \text{Cl}_{C_\beta} G_\beta(A, B) \). For each \( \alpha \in \Lambda \),

\[
K_\beta \cap C_\alpha = \text{Cl}_{C_\alpha} G_\alpha(A \cap C_\alpha, B \cap C_\alpha),
\]

since \( \{(C_\alpha, G_\alpha) : \alpha \in \Lambda \} \) is linearly ordered. Hence \( K_\beta \) is closed in \( C_\beta \) and \( K_\beta = \text{Cl}_{C_\beta} G_\beta(A, B) \). Furthermore, since

\[
\text{Cl}_{C_\alpha} G_\alpha(A \cap C_\alpha, B \cap C_\alpha) \subseteq C_\alpha - B \cap C_\alpha,
\]

it holds that \( K_\beta \subseteq C_\beta - B \). Thus \( G_\beta \) is a monotone normality operator for \( C_\beta \).

By Zorn’s Lemma, let \((C_0, G_0)\) be a maximal element of \( \mathcal{G} \). To complete the proof we need only show that \( C_0 = B \). Suppose not. Then there exists \( E \in B - C_0 \). Let \( C_1 = C_0 \cup \{E\} \). Now \( C_0 \) and \( E \) are closed monotonically normal subspaces of \( C_1 = C_0 \cup E \), and hence \( C_1 \) is monotonically normal by Theorem 1.1. Thus, by the proof of Theorem 1.1, one may obtain a monotone normality operator \( G_1 \) of \( C_1 \) such that, for each pair \((A, B)\) of separated subsets of \( C_1 \), \( G_1(A, B) \cap C_0 = G_0(A \cap C_0, B \cap C_0) \). Consequently, \((C_0, G_0) \prec (C_1, G_1)\), contradicting the maximality of \((C_0, G_0)\). Hence \( C_0 = B \), and \( X \) is monotonically normal.

REFERENCES