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ADJUNCTION SPACES OF MONOTONICALLY NORMAL SPACES AND SPACES DOMINATED BY MONOTONICALLY NORMAL SUBSETS

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ABSTRACT. In this paper, we shall prove the following results: (1) the adjunction space of two monotonically normal spaces is also monotonically normal, (2) a topological space is monotonically normal if and only if it is dominated by a collection of monotonically normal subsets.

The class of monotonically normal spaces was introduced by P. Zenor, and studied by P. Zenor, R. Heath and D. Lutzer [5], C. R. Borges [2, 3] and others. In this paper, we shall first prove that the adjunction space of two monotonically normal spaces is monotonically normal, and as an immediate corollary that a topological space is an AR($\mathcal{MN}$) (resp. ANR($\mathcal{MN}$)) if and only if it is an AE($\mathcal{MN}$) (resp. ANE($\mathcal{MN}$)), where $\mathcal{MN}$ is the class of all monotonically normal spaces.

In [5, p. 483], it was shown that if a topological space X can be covered by a locally finite (or even hereditarily closure preserving) collection of closed monotonically normal subspaces, then X is monotonically normal. Concerning this result, it was asked whether a topological space X must be monotonically normal provided that X is dominated by a collection such subspaces. In §2, we shall answer this question affirmatively.

Throughout this paper, all spaces are assumed to be Hausdorff topological spaces. Cl_X denotes the closure operator in a space X.

1. Adjunction spaces. In this paper, as the definition of monotonically normal space, we exclusively use [5, Lemma 2.2(a)]; i.e. a space X is monotonically normal if there is a function G which assigns to each pair (A, B) of separated subsets of X an open set $G(A, B)$ satisfying

(i) $A \subset G(A, B) \subset \mathrm{Cl}_X G(A, B) \subset X - B$,

(ii) if (A', B') is a pair of separated subsets having $A \subset A'$ and $B \supset B'$, then $G(A, B) \subset G(A', B')$, where two subsets A, B of X are separated if $A \cap \mathrm{Cl}_X B = \varnothing$, $\mathrm{Cl}_X A \cap B = \varnothing$.

The function G is called a monotone normality operator for X. We put $G(\varnothing, B) = \varnothing$, $G(A, \varnothing) = X$, where $A \neq \varnothing$. These are used in the proof of Theorem 1.1; for instance, $G_2(A_Y, B_Y)$, $G_1(A_1, B_1)$ if $A_Y = \varnothing$, $B_Y = \varnothing$, $A_1 = \varnothing$ or $B_1 = \varnothing$.

THEOREM 1.1. *Let X and Y be monotonically normal spaces, F a closed subset of X and $f\colon F \to Y$ a continuous mapping. Then the adjunction space $X \cup_f Y = Z$ is monotonically normal.*

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PROOF. Let $h\colon X \to Z$, $k\colon Y \to Z$ be the natural projections. By definition of $X \cup_f Y$, $U \subset Z$ is open (closed) if and only if $h^{-1}(U)$ and $k^{-1}(U)$ are open (closed); furthermore, k and $h|X - F$ are homeomorphisms into. For convenience, for a subset A of Z, we let $A_X = h^{-1}(A)$, $A_Y = k^{-1}(A)$.

Let G_1 and G_2 be monotone normality operators for X and Y, respectively, and (A, B) a pair of separated subsets of Z. We shall show that there is a monotone normality operator G for Z such that $k^{-1}(G(A,B)) = G_2(A_Y, B_Y)$.

For a pair (A, B) of separated subsets of Z, (A_Y, B_Y) is a pair of separated subsets of Y. Let

$$A_1 = A_X \cup f^{-1}(G_2(A_Y, B_Y)),$$
$$B_1 = (F - f^{-1}(\mathrm{Cl}_Y\, G_2(A_Y, B_Y))) \cup B_X.$$

Then (A_1, B_1) is a pair of separated subsets of X. Since there is an open subset $G_1(A_1, B_1)$ of X such that

$$A_1 \subset G_1(A_1, B_1) \subset \mathrm{Cl}_X\, G_1(A_1, B_1) \subset X - B_1,$$

there is an open subset U_A of X such that

$$U_A - F = G_1(A_1, B_1) - F, \quad U_A \cap F = f^{-1}(G_2(A_Y, B_Y)).$$

Let $G(A,B) = h(U_A) \cup k(G_2(A_Y, B_Y))$. Then it is easily seen that G is a monotone normality operator for Z such that $k^{-1}(G(A,B)) = G_2(A_Y, B_Y)$. This completes the proof.

AR($\mathcal{C}$) (resp. ANR($\mathcal{C}$)) is the abbreviation for absolute (resp. neighborhood) retract for the class $\mathcal{C}$ and AE($\mathcal{C}$) (resp. ANE($\mathcal{C}$)) the abbreviation for absolute (resp. neighborhood) extensor for the class $\mathcal{C}$. For these definitions, see [**6**]. Note that, in [**4**], ES($\mathcal{C}$) and NES($\mathcal{C}$) are used instead of AE($\mathcal{C}$) and ANE($\mathcal{C}$), respectively.

COROLLARY 1.2. *Let X be a monotonically normal space. Then X is an* AR($\mathcal{MN}$) *(resp.* ANR($\mathcal{MN}$)*) if and only if X is an* AE($\mathcal{MN}$) *(resp.* ANE($\mathcal{MN}$)*).*

PROOF. This follows from Theorem 1.1, using the same method of proof of Theorem 8.1 in [**4**].

2. Spaces dominated by monotonically normal subsets. We start by reproducing Definition 8.1 in [**7**].

DEFINITION 2.1. Let X be a space, and $\mathcal{B}$ a collection of closed subsets of X. Then $\mathcal{B}$ dominates X if, whenever $A \subset X$ has a closed intersection with every element of some subcollection $\mathcal{B}_1$ of $\mathcal{B}$ which covers A, then A is closed.

In [**7**] (resp. [**1**]), it is shown that a space is paracompact (resp. stratifiable) if and only if it is dominated by a collection of paracompact (resp. stratifiable) subspaces. We prove the following:

THEOREM 2.2. *A space is monotonically normal if and only if it is dominated by a collection of monotonically normal subsets.*

PROOF. Since the "only if" part is trivial, we prove the "if" part. Let $\mathcal{B}$ be a dominating collection of monotonically normal subsets of a space X. Consider the class $\mathcal{G}$ of all pairs of the form $(\mathcal{C}_\alpha, G_\alpha)$, where $\mathcal{C}_\alpha \subset \mathcal{B}$, and G_α is a monotone normality operator for $C_\alpha = \bigcup \mathcal{C}_\alpha$. (Throughout this proof, $\bigcup \mathcal{C}_\gamma$ will be denoted by C_γ for any $\mathcal{C}_\gamma \subset \mathcal{B}$.) We partially order $\mathcal{G}$ by letting $(\mathcal{C}_\alpha, G_\alpha) \le (\mathcal{C}_\beta, G_\beta)$ whenever $\mathcal{C}_\alpha \subset \mathcal{C}_\beta$ and, for each pair (A,B) of separated subsets of C_β, $G_\beta(A,B) \cap C_\alpha = G_\alpha(A \cap C_\alpha, B \cap C_\alpha)$.

We now show that any linearly ordered subfamily $\{(\mathcal{C}_\alpha, G_\alpha)\colon \alpha\in\Lambda\}$ of $\mathcal{G}$ has an upper bound $(\mathcal{C}_\beta, G_\beta)$. Let $\mathcal{C}_\beta = \bigcup\{\mathcal{C}_\alpha\colon \alpha\in\Lambda\}$. For each pair (A,B) of separated subsets of C_β, let

$$G_\beta(A,B) = \bigcup\{G_\alpha(A\cap C_\alpha, B\cap C_\alpha)\colon \alpha\in\Lambda\},$$

and let us show that G_β is a monotone normality operator for C_β. In fact, first, since $G_\beta(A,B)\cap C_\alpha = G_\alpha(A\cap C_\alpha, B\cap C_\alpha)$ for each $\alpha\in\Lambda$, $G_\beta(A,B)$ is open in C_β. Next, let

$$K_\beta = \bigcup\{\mathrm{Cl}_{C_\alpha} G_\alpha(A\cap C_\alpha, B\cap C_\alpha)\colon \alpha\in\Lambda\}.$$

Clearly $G_\beta(A,B)\subset K_\beta\subset \mathrm{Cl}_{C_\beta} G_\beta(A,B)$. For each $\alpha\in\Lambda$,

$$K_\beta\cap C_\alpha = \mathrm{Cl}_{C_\alpha} G_\alpha(A\cap C_\alpha, B\cap C_\alpha),$$

since $\{(\mathcal{C}_\alpha, G_\alpha)\colon \alpha\in\Lambda\}$ is linearly ordered. Hence K_β is closed in C_β and $K_\beta = \mathrm{Cl}_{C_\beta} G_\beta(A,B)$. Furthermore, since

$$\mathrm{Cl}_{C_\alpha} G_\alpha(A\cap C_\alpha, B\cap C_\alpha)\subset C_\alpha - B\cap C_\alpha,$$

it holds that $K_\beta\subset C_\beta - B$. Thus G_β is a monotone normality operator for C_β.

By Zorn's Lemma, let $(\mathcal{C}_0, G_0)$ be a maximal element of $\mathcal{G}$. To complete the proof we need only show that $\mathcal{C}_0 = \mathcal{B}$. Suppose not. Then there exists $E\in\mathcal{B}-\mathcal{C}_0$. Let $\mathcal{C}_1 = \mathcal{C}_0\cup\{E\}$. Now C_0 and E are closed monotonically normal subspaces of $C_1 = C_0\cup E$, and hence C_1 is monotonically normal by Theorem 1.1. Thus, by the proof of Theorem 1.1, one may obtain a monotone normality operator G_1 of C_1 such that, for each pair (A,B) of separated subsets of C_1, $G_1(A,B)\cap C_0 = G_0(A\cap C_0, B\cap C_0)$. Consequently, $(\mathcal{C}_0, G_0) < (\mathcal{C}_1, G_1)$, contradicting the maximality of $(\mathcal{C}_0, G_0)$. Hence $\mathcal{C}_0 = \mathcal{B}$, and X is monotonically normal.

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