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STABLE HOMOTOPY TYPES OF STUNTED REAL PROJECTIVE SPACES

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ABSTRACT. The purpose of this note is to improve partly necessary conditions for two stunted real projective spaces to be of the same stable homotopy type given in [4].

1. Introduction. Let RP^n be the n-dimensional real projective space. If $k \geqslant 0$, we have natural inclusion $RP^{n-1} \subset RP^{n+k}$, and denote by RP_n^{n+k} the quotient space RP^{n+k}/RP^{n-1}. S. Feder, S. Gitler and M. E. Mahowald in [4] have obtained necessary and sufficient conditions for the stunted real projective spaces RP_n^{n+k} and RP_m^{m+k} to be of the same stable homotopy type for a great number of values of n, m and k. Let $\phi(k)$ be the number of integers s such that $0 < s \leqslant k$ and $s \equiv 0, 1, 2$ or $4 \pmod 8$, and let $A_k = 2^{\phi(k)}$.

THEOREM 1. *Let k and n be integers such that $n \equiv 0 \pmod{A_k/2}$. If $n \not\equiv m \pmod{A_k}$, then the spaces RP_n^{n+k} and RP_m^{m+k} are not of the same stable homotopy type.*

Theorem 1 partly improves Theorem 1.1 in [4].[2] For the converse of the result, it is well known (cf. [1–3]) that if $n \equiv m \pmod{A_k}$, then the spaces RP_n^{n+k} and RP_m^{m+k} are of the same stable homotopy type.

2. Proof of Theorem 1. In case $k \leqslant 8$, we can prove without difficulty that if RP_n^{n+k} and RP_m^{m+k} are of the same stable homotopy type, then $n \equiv m \pmod{A_k}$, using naturality of the squaring operations.

Consider the case $k > 8$. Suppose $m < n$ and there is a homotopy equivalence f: $S^t RP_n^{n+k} \to S^{t+n-m} RP_m^{m+k}$ for some integer $t \geqslant 0$. Then $n \equiv m \pmod{A_k/2}$, by Theorem 1.1 in [4] (cf. also [5 and 6]). We may assume $t \equiv 0 \pmod 8$, and so $t + n - m \equiv 0 \pmod 8$. Put $t = 8u$ and $t + n - m = 8v$ and let

$$I^u\colon \widetilde{KO}(RP_n^{n+k}) \to \widetilde{KO}(S^t RP_n^{n+k}), \qquad I^v\colon \widetilde{KO}(RP_m^{m+k}) \to \widetilde{KO}(S^{8v} RP_m^{m+k})$$

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[2]Theorem 1 was known to the authors of [4], but the result was not published.

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be the isomorphisms defined by the Bott periodicity. If $l \equiv 0 \pmod 4$, we have by [**1**, Theorem 7.4],

$$\widetilde{KO}(RP_l^{l+k}) \cong Z \oplus \widetilde{KO}(RP_{l+1}^{l+k}) \cong Z \oplus Z_{A_k}.$$

Denote by $\nu^{(l)}$ and $\lambda^{(l+1)}$ the generators of the first summand and the second summand, respectively. We may put

$$f^*I^v\nu^{(m)} = \varepsilon I^u\nu^{(n)} + aI^u\lambda^{(n+1)}, \qquad f^*I^v\lambda^{(m+1)} = (2b+\varepsilon)I^u\lambda^{(n+1)},$$

where a and b are some integers and $\varepsilon = \pm 1$. Let Ψ^3 be the third Adams operation. Then by [**1**, Theorem 5.1, Corollary 5.3 and Theorem 7.4], we have

$$\begin{aligned}
\Psi^3 f^*I^v\nu^{(m)} &= f^*\Psi^3 I^v\nu^{(m)} = 3^{4v}f^*I^v\Psi^3\nu^{(m)} \\
&= 3^{4v}f^*I^v\{3^{m/2}\nu^{(m)} + 2^{-1}(3^{m/2}-1)\lambda^{(m+1)}\} \\
&= 3^{4v}\{3^{m/2}(\varepsilon I^u\nu^{(n)} + aI^u\lambda^{(n+1)}) + 2^{-1}(3^{m/2}-1)(2b+\varepsilon)I^u\lambda^{(n+1)}\} \\
&= \varepsilon 3^{4u+n/2}I^u\nu^{(n)} \\
&\quad + 3^{4u}\left[3^{n/2}a + 3^{(n-m)/2}\{(3^{m/2}-1)b + 2^{-1}(3^{m/2}-1)\varepsilon\}\right]I^u\lambda^{(n+1)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Psi^3 f^*I^v\nu^{(m)} &= \varepsilon\Psi^3 I^u\nu^{(n)} + a\Psi^3 I^u\lambda^{(n+1)} = \varepsilon 3^{4u}I^u\Psi^3\nu^{(n)} + a3^{4u}I^u\Psi^3\lambda^{(n+1)} \\
&= \varepsilon 3^{4u}I^u\{3^{n/2}\nu^{(n)} + 2^{-1}(3^{n/2}-1)\lambda^{(n+1)}\} + 3^{4u}aI^u\lambda^{(n+1)} \\
&= \varepsilon 3^{4u+n/2}I^u\nu^{(n)} + 3^{4u}\{2^{-1}(3^{n/2}-1)\varepsilon + a\}I^u\lambda^{(n+1)}.
\end{aligned}$$

We have, therefore,

$$3^{n/2}a + 3^{(n-m)/2}\{(3^{m/2}-1)b + 2^{-1}(3^{m/2}-1)\varepsilon\} \equiv 2^{-1}(3^{n/2}-1)\varepsilon + a \pmod{A_k},$$

because the order of the element $I^u\lambda^{(n+1)}$ is equal to $A_k = 2^{\phi(k)}$. Since $n \equiv m \equiv 0 \pmod{A_k/2}$, $3^{n/2}-1 \equiv 3^{m/2}-1 \equiv 0 \pmod{A_k}$ by [**1**, Lemma 8.1]. Hence we obtain $3^{(n-m)/2}-1 \equiv 0 \pmod{2A_k}$, and so we get $n - m \equiv 0 \pmod{A_k}$ by [**1**, Lemma 8.1].

3. Remark. Combining Theorem 1 with known results, we have

COROLLARY 2. *Let $k \leqslant 8$, or $k > 8$ and $n \equiv 0 \pmod{A_k/2}$. Then RP_n^{n+k} and RP_m^{m+k} are of the same stable homotopy type if and only if $n \equiv m \pmod{A_k}$.*

REFERENCES

1. J. F. Adams, *Vector fields on spheres*, Ann. of Math. (2) **75** (1962), 603–632.
2. _____, *On the groups $J(X)$*. II, Topology **3** (1965), 137–171.
3. M. F. Atiyah, *Thom complexes*, Proc. London Math. Soc. **11** (1961), 291–310.
4. S. Feder, S. Gitler and M. E. Mahowald, *On the stable homotopy type of stunted projective spaces*, Bol. Soc. Mat. Mexicana **22** (1977), 1–5.
5. T. Kobayashi, *On the odd order nonsingular immersions of real projective spaces*, J. Sci. Hiroshima Univ. Ser. A–I **33** (1969), 197–207.
6. T. Kobayashi and M. Sugawara, *On stable homotopy types of stunted lens spaces*, Hiroshima Math. J. **1** (1971), 287–304.

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