STABLE HOMOTOPY TYPES
OF STUNTED REAL PROJECTIVE SPACES

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Abstract. The purpose of this note is to improve partly necessary conditions for two stunted real projective spaces to be of the same stable homotopy type given in [4].

1. Introduction. Let $RP^n$ be the $n$-dimensional real projective space. If $k \geq 0$, we have natural inclusion $RP^{n-1} \subset RP^{n+k}$, and denote by $RP^{n+k}$ the quotient space $RP^{n+k}/RP^{n-1}$. S. Feder, S. Gitler and M. E. Mahowald in [4] have obtained necessary and sufficient conditions for the stunted real projective spaces $RP^{n+k}$ and $RP^{m+k}$ to be of the same stable homotopy type for a great number of values of $n$, $m$ and $k$. Let $\phi(k)$ be the number of integers $s$ such that $0 < s < k$ and $s \equiv 0, 1, 2$ or $4$ (mod 8), and let $A_k = 2^{\phi(k)}$.

Theorem 1. Let $k$ and $n$ be integers such that $n \equiv 0$ (mod $A_k/2$). If $n \equiv m$ (mod $A_k$), then the spaces $RP^{n+k}$ and $RP^{m+k}$ are not of the same stable homotopy type.

Theorem 1 partly improves Theorem 1.1 in [4]. For the converse of the result, it is well known (cf. [1–3]) that if $n \equiv m$ (mod $A_k$), then the spaces $RP^{n+k}$ and $RP^{m+k}$ are of the same stable homotopy type.

2. Proof of Theorem 1. In case $k \leq 8$, we can prove without difficulty that if $RP^{n+k}$ and $RP^{m+k}$ are of the same stable homotopy type, then $n \equiv m$ (mod $A_k$), using naturality of the squaring operations.

Consider the case $k > 8$. Suppose $m < n$ and there is a homotopy equivalence $f$: $S^t RP^{n+k} \to S^{t+n-m} RP^{m+k}$ for some integer $t \geq 0$. Then $n \equiv m$ (mod $A_k/2$), by Theorem 1.1 in [4] (cf. also [5 and 6]). We may assume $t \equiv 0$ (mod 8), and so $t + n - m \equiv 0$ (mod 8). Put $t = 8u$ and $t + n - m = 8v$ and let

$I^u: \widetilde{KO}(RP^{n+k}) \to \widetilde{KO}(S^t RP^{n+k}), \quad I^v: \widetilde{KO}(RP^{m+k}) \to \widetilde{KO}(S^{8v}RP^{m+k})$
be the isomorphisms defined by the Bott periodicity. If \( l \equiv 0 \pmod{4} \), we have by [1, Theorem 7.4],

\[
\overline{KO}(RP^{l+k}) = \mathbb{Z} \oplus \overline{KO}(RP^{l+1}) = \mathbb{Z} \oplus \mathbb{Z}_{A_k}.
\]

Denote by \( \nu^{(l)} \) and \( \lambda^{(l+1)} \) the generators of the first summand and the second summand, respectively. We may put

\[
f^*I^v\nu^{(m)} = \epsilon I^u\nu^{(n)} + a I^u\lambda^{(n+1)}, \quad f^*I^v\lambda^{(m+1)} = (2b + \epsilon) I^u\lambda^{(n+1)},
\]

where \( a \) and \( b \) are some integers and \( \epsilon = \pm 1 \). Let \( \Psi^3 \) be the third Adams operation. Then by [1, Theorem 5.1, Corollary 5.3 and Theorem 7.4], we have

\[
\Psi^3 f^*I^v\nu^{(m)} = f^*\Psi^3 I^v\nu^{(m)} = 3^4 I^v\nu^{(m)} = 3^4 (3^{m/2} \nu^{(m)} + 2^{-1}(3^{m/2} - 1)\lambda^{(m+1)})
\]

\[
= 3^4 (3^{m/2} (\epsilon I^u\nu^{(n)} + a I^u\lambda^{(n+1)}) + 2^{-1}(3^{m/2} - 1)(2b + \epsilon) I^u\lambda^{(n+1)})
\]

\[
= \epsilon 3^4 u^{n/2} I^v\nu^{(n)} + 3^4 u^{3/2} (3^{m/2} - 1) b + 2^{-1}(3^{m/2} - 1)\epsilon I^u\lambda^{(n+1)}.
\]

On the other hand,

\[
\Psi^3 f^*I^v\nu^{(m)} = \epsilon \Psi^3 I^u\nu^{(n)} + a \Psi^3 I^u\lambda^{(n+1)} = \epsilon 3^4 I^u\nu^{(n)} + a 3^4 I^u\nu^{(n+1)}
\]

\[
= \epsilon 3^4 I^u (3^{n/2} \nu^{(n)} + 2^{-1}(3^{n/2} - 1)\lambda^{(n+1)}) + 3^4 a I^u\lambda^{(n+1)}
\]

\[
= \epsilon 3^4 u^{n/2} I^v\nu^{(n)} + 3^4 u^{3/2} (2^{-1}(3^{n/2} - 1)\epsilon + a) I^u\lambda^{(n+1)}.
\]

We have, therefore,

\[
3^{n/2} a + 3^{(n-m)/2} (3^{m/2} - 1) b + 2^{-1}(3^{m/2} - 1)\epsilon \equiv 2^{-1}(3^{n/2} - 1)\epsilon + a
\]

\[(\text{mod } A_k),\]

because the order of the element \( I^u\lambda^{(n+1)} \) is equal to \( A_k = 2^k \). Since \( n \equiv m \equiv 0 \pmod{A_k/2} \), \( 3^{n/2} - 1 \equiv 3^{m/2} - 1 \equiv 0 \pmod{A_k} \) by [1, Lemma 8.1]. Hence we obtain \( 3^{(n-m)/2} - 1 \equiv 0 \pmod{2 A_k} \), and so we get \( n - m \equiv 0 \pmod{A_k} \) by [1, Lemma 8.1].

3. Remark. Combining Theorem 1 with known results, we have

**Corollary 2.** Let \( k \leq 8 \), or \( k > 8 \) and \( n \equiv 0 \pmod{A_k/2} \). Then \( RP^{n+k} \) and \( RP^{m+k} \) are of the same stable homotopy type if and only if \( n \equiv m \pmod{A_k} \).

**References**


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