PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 87, Number 4, April 1983

A NOTE ON THE COHEN-MACAULAY TYPE OF LINES IN UNIFORM POSITION IN A^{n+1}

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ABSTRACT. Let $\mathcal{L}_1,\dots,\mathcal{L}_s$ be s-distinct lines in $\mathbf{A}_k^{n+1}$ passing through the origin. Assume $s=\binom{n+d}{n}-\lambda$ where $n, d \geq 2$. If $\mathcal{L}_1,\dots,\mathcal{L}_s$ are in generic s-position, and $\lambda=0,1,\dots,n-1$, then the Cohen-Macaulay type, $t(\mathcal{L}_1,\dots,\mathcal{L}_s)$, of $\mathcal{L}_1,\dots,\mathcal{L}_s$ is given by the following formula: $t(\mathcal{L}_1,\dots,\mathcal{L}_s)=\binom{n+d-1}{n-1}-\lambda$. This formula is known to be false for $\lambda=n$. In this paper, we show that if $\mathcal{L}_1,\dots,\mathcal{L}_s$ are in uniform position, and $\lambda=n$, then $t(\mathcal{L}_1,\dots,\mathcal{L}_s)=\binom{n+d-1}{n-1}-n$.

Introduction. Throughout this paper, k will denote an algebraically closed field of arbitrary characteristic. We shall let $\mathbf{A}_k^{n+1}$ and $\mathbf{P}_k^n$ denote affine $(n+1)$-space and projective n-space over k respectively.

If $\mathcal{L}_1,\dots,\mathcal{L}_s$ are s-distinct lines in $\mathbf{A}_k^{n+1}$ passing through the origin, we shall identify $\mathcal{L}_i$ with its direction numbers $P_i \in P_k^n$, $i=1,\dots,s$. Let $\mathcal{S}=k[X_0,\dots,X_n]$ denote the coordinate ring of $\mathbf{A}_k^{n+1}$ (or $\mathbf{P}_k^n$). The jth homogeneous piece of $\mathcal{S}$ will be denoted by $\mathcal{S}_j$. Thus, $\mathcal{S}_j$ is a k-vector space of dimension $\binom{n+j}{n}$. Set $\nu(j)=\binom{n+j}{n}$. Let $\{g_{1j},\dots,g_{\nu(j)j}\}$ be a vector space basis of $\mathcal{S}_j$. We can then form an $[s\times\nu(j)]$-matrix $G_j(\mathcal{L}_1,\dots,\mathcal{L}_s)$ as follows: the (α,β)th entry of $G_j(\mathcal{L}_1,\dots,\mathcal{L}_s)$ is $g_{\beta j}(P_\alpha)$. We shall denote the rank of the matrix $G_j(\mathcal{L}_1,\dots,\mathcal{L}_s)$ by $\operatorname{rk}(G_j(\mathcal{L}_1,\dots,\mathcal{L}_s))$.

The lines $\mathcal{L}_1,\dots,\mathcal{L}_s$ are said to be in generic s-position if $\operatorname{rk}(G_j(\mathcal{L}_1,\dots,\mathcal{L}_s))=\min\{s,\nu(j)\}$ for all $j\geq 1$. If $0<t\leq s$, we say $\mathcal{L}_1,\dots,\mathcal{L}_s$ are in generic t-position if every t-subset of $\{\mathcal{L}_1,\dots,\mathcal{L}_s\}$ is in generic t-position. Finally, the set of lines $\mathcal{L}_1,\dots,\mathcal{L}_s$ is said to be in uniform position if it is in generic t-position for every $0<t\leq s$.

Now let $\mathcal{L}_1,\dots,\mathcal{L}_s$ be in generic s-position in $\mathbf{A}_k^{n+1}$. We assume $s=\nu(d)-\lambda$. Here $0\leq\lambda\leq\nu(d-1)+1$. To avoid trivial cases, we shall assume throughout the rest of this paper that $n\geq 2$, and $d\geq 2$. By the Cohen-Macaulay type, $t(\mathcal{L}_1,\dots,\mathcal{L}_s)$, of $\mathcal{L}_1,\dots,\mathcal{L}_s$, we shall mean the Cohen-Macaulay type of the corresponding local ring (at the origin) of the reduced curve $\bigcup_{i=1}^s\mathcal{L}_i$. The reader is referred to [2] for all pertinent definitions not given here.

$t(\mathcal{L}_1,\dots,\mathcal{L}_s)$ is known when $\lambda=0,1,\dots,n-1$. A precise formula is given as follows:

$$t(\mathcal{L}_1,\dots,\mathcal{L}_s)=\binom{n+d-1}{n-1}-\lambda,\qquad \lambda=0,1,\dots,n-1. \tag{1}$$

Received by the editors August 11, 1982 and, in revised form, September 3, 1982.
1980 *Mathematics Subject Classification*. Primary 13H10, 14H20, 13H15.
Key words and phrases. Cohen-Macaulay type, generic s-position, uniform position.

When $\lambda = 0$, equation (1) follows from Proposition 16 in [3]. For $\lambda = 1, 2, \ldots, n-1$, equation (1) was established by L. Roberts in [5].

When $\lambda = n$, Roberts showed [5] that equation (1) continues to hold "generically". More precisely, let us topologize the lines through the origin in $\mathbf{A}_k^{n+1}$ by identifying $\mathcal{L}_1, \ldots, \mathcal{L}_s$ with the point $(P_1, \ldots, P_s)$ in s-copies $\mathbf{P}_k^n \times \cdots \times \mathbf{P}_k^n$ of $\mathbf{P}_k^n$. Then, when $\lambda = n$, Roberts showed that there exists a nonempty, Zariski open subset U of $\mathbf{P}_k^n \times \cdots \times \mathbf{P}_k^n$ ($s = (\nu(d) - n)$-copies here) such that $t(\mathcal{L}_1, \ldots, \mathcal{L}_s) = \binom{n+d-1}{n-1} - n$ whenever $(\mathcal{L}_1, \ldots, \mathcal{L}_s) \in U$.

The exact nature of the open set U is not clear. For instance, $U' = \{(\mathcal{L}_1, \ldots, \mathcal{L}_s) \in \mathbf{P}_k^n \times \cdots \times \mathbf{P}_k^n \mid \mathcal{L}_1, \ldots, \mathcal{L}_s$ are in generic s-position in $\mathbf{A}_k^{n+1}\}$ is easily seen to be a nonempty, open subset of $\mathbf{P}_k^n \times \cdots \times \mathbf{P}_k^n$. But, for $\lambda = n$, equation (1) is not true on U'. Concrete examples can be found in [1]. In this paper, we give a specific open set upon which equation (1) continues to hold when $\lambda = n$. Namely, let $U'' = \{(\mathcal{L}_1, \ldots, \mathcal{L}_s) \in \mathbf{P}_k^n \times \cdots \times \mathbf{P}_k^n \mid \mathcal{L}_1, \ldots, \mathcal{L}_s$ are in uniform position$\}$. One easily checks that U'' is a nonempty, open subset of $\mathbf{P}_k^n \times \cdots \times \mathbf{P}_k^n$ contained in U'. We shall show that the Cohen-Macaulay type of $\mathcal{L}_1, \ldots, \mathcal{L}_s$ is given by equation (1) when $\lambda = n$, and $(\mathcal{L}_1, \ldots, \mathcal{L}_s) \in U''$.

The question of the "generic" Cohen-Macaulay type of lines in $\mathbf{A}_k^3$ (i.e. $n = 2$) has been completely solved in [4]. The results there seem to indicate that "uniform position" is not a strong enough hypothesis to give an equation like (1) when $n > 2$, and $\lambda > n$.

Finally, the author assumes the reader is familiar with the contents of [2, 3 and 4] and will use results from these three papers freely.

Main result.

THEOREM. *Let $\mathcal{L}_1, \ldots, \mathcal{L}_s$ be s-distinct lines passing through the origin in $\mathbf{A}_k^{n+1}$. We assume $s = \binom{n+d}{n} - n$ with $n, d \geqslant 2$. If $\mathcal{L}_1, \ldots, \mathcal{L}_s$ are in uniform position, then*

$$t(\mathcal{L}_1, \ldots, \mathcal{L}_s) = \binom{n+d-1}{n-1} - n. \tag{2}$$

PROOF. Recall $\mathcal{S} = k[X_0, \ldots, X_n]$. Let $\mathcal{I}$ denote the ideal of $\cup_{i=1}^s \mathcal{L}_i$ in $\mathcal{S}$. Thus, $\mathcal{I}$ is a homogeneous, unmixed, radical ideal of height n in $\mathcal{S}$. We first argue that we can assume without loss of generality that every nonzero form in $\mathcal{I}_d$ is irreducible. Here, $\mathcal{I}_d$ denotes the dth homogeneous piece of $\mathcal{I}$.

We first consider the special case $d = 2$. Then $s = \binom{n+2}{n} - n$ with $n \geqslant 2$. Clearly, $2 = \min\{j \mid \binom{n+j}{j} > s\}$ and, thus, it follows from [2, Remark (3), p. 8] that $\mathcal{I}_2 \neq (0)$. Suppose $\mathcal{I}_2$ contains a nonzero, reducible quadratic form F. Then $F = l_1 l_2$ with l_i a one form in $\mathcal{S}$. $\{P_1, \ldots, P_s\} \subseteq V(F) = V(l_1) \cup V(l_2)$ in P_k^n. Suppose $\{P_1, \ldots, P_s\} \subseteq V(l_1)$. Again using [2, Remark (3), p. 8], we would have $\nu(1) - 1 \geqslant s$. This last inequality implies $4n \geqslant (n+2)(n+1)$ which is impossible since $n \geqslant 2$. Thus, some of the P_i lie on l_1 and some on l_2. Let t_i denote the number of points in $\{P_1, \ldots, P_s\} \cap V(l_i)$, $i = 1, 2$. Then $t_i \geqslant 1$, and $t_1 + t_2 \geqslant s$. Again applying [2, Remark (3), p. 8], we have the following inequality:

$$2[\nu(1) - 1] \geqslant t_1 + t_2 \geqslant s. \tag{3}$$

The inequality in (3) implies $6n \geqslant n^2 + 3n + 2$. This is possible only in the case $n = 2$. But then $s = 4$, and we have 4-lines in uniform position in $\mathbf{A}_k^3$. Now it is well known (e.g. [**2**, Theorem 7]) that 4-lines in uniform position in $\mathbf{A}_k^3$ have a Gorenstein singularity at the origin. Therefore, $t(\mathfrak{L}_1, \mathfrak{L}_2, \mathfrak{L}_3, \mathfrak{L}_4) = 1 = \binom{2+2-1}{2-1} - 2$, and our theorem is correct in this case. So, when $d = 2$, we can assume $n \geqslant 3$. In this case, the above argument shows any $F \in \mathfrak{I}_2 - (0)$ is irreducible.

Now assume $d \geqslant 3$. We first note that $\nu(d-1) < s < \nu(d)$ (these inequalities hold for $d = 2$ as well). In particular, $d = \min\{j \mid \nu(j) > s\}$. Thus, $\mathfrak{I}_j = 0$ for $j < d$, and $\mathfrak{I}_d \neq 0$. Since $d \geqslant 3$, we have $\nu(d-1) + n \leqslant s$. Thus, if we write $s = \nu(d-1) + h$, then $n \leqslant h < \binom{n+d-1}{n-1}$. It now follows immediately from [**4**, Theorem 3.4] that every $F \in \mathfrak{I}_d - (0)$ is irreducible. This completes all cases for d, and, so, henceforth, we shall assume that any nonzero form in $\mathfrak{I}_d$ is irreducible.

Since $\mathfrak{L}_1, \ldots, \mathfrak{L}_s$ are in generic s-position, $\dim_k\{\mathfrak{I}_j\} = \nu(j) - \min\{s, \nu(j)\}$ for $j = 1, 2 \ldots$. In particular, $\dim \mathfrak{I}_d = n$. Thus, $\mathfrak{I}_d = \oplus_{i=1}^n kF_i$ with $F_1, \ldots, F_n$ irreducible, d-forms in $\mathfrak{I}$.

Let the associated primes of $\mathfrak{I}$ be denoted by $\mathfrak{P}_1, \ldots, \mathfrak{P}_s$, and let the associated primes of the ideal (F_i, F_j) be denoted by $Q_1^{i,j}, \ldots, Q_{r(i,j)}^{i,j}$. Here $1 \leqslant i \neq j \leqslant n$, and $r(i, j)$ is some positive integer. Since $\mathfrak{S}$ is a Cohen-Macaulay ring, the grade of $\mathfrak{I}$ is n. Since $\{F_i, F_j\}$ are linearly independent, irreducible forms, the grade of (F_i, F_j) is two. In particular, the maximal ideal $(X_0, \ldots, X_n)$ properly contains any $\mathfrak{P}_i$ or $Q_t^{i,j}$. Thus, if we set

$$W = \left\{ \bigcup_{i=1}^{s} \mathfrak{P}_i \cap \mathfrak{S}_1 \right\} \cup \left\{ \bigcup_{i \neq j} \left(\bigcup_{t} Q_t^{i,j} \cap \mathfrak{S}_1 \right) \right\},$$

then $\mathfrak{S}_1 - W$ is nonempty. Let $g \in \mathfrak{S}_1 - W$. Then g is not in any associated prime of $\mathfrak{I}$ or of (F_i, F_j) for $i \neq j$. Changing coordinates in $\mathbf{P}_k^n$ if need be, we can assume $g = X_0$ without any loss in generality.

Now set $\mathfrak{R} = \mathfrak{S}/\mathfrak{I}$, the coordinate ring of $\cup_{i=1}^s \mathfrak{L}_i$. Let x_0 denote the image of X_0 in $\mathfrak{R}$. Then x_0 is a nonzero-divisor in $\mathfrak{R}$. Also, for any $1 \leqslant i \neq j \leqslant n$, $\{F_i, F_j, X_0\}$ is a regular sequence of length three in $\mathfrak{S}$. Since F_i, F_j and X_0 are all forms, $\{X_0, F_i, F_j\}$ is also a regular sequence of length three in $\mathfrak{S}$.

Set $\mathfrak{T} = k[Y_1, \ldots, Y_n]$. Here $Y_1, \ldots, Y_n$ are indeterminates over k. Let $\psi\colon \mathfrak{S} \to \mathfrak{T}$ be the k-algebra epimorphism given by $\psi(X_0) = 0$, $\psi(X_i) = Y_i$ for $i = 1, \ldots, n$. Then we have a natural k-algebra epimorphism $\sigma\colon \mathfrak{T} \to \mathfrak{R}/x_0\mathfrak{R}$ such that the kernel of σ is $\psi(\mathfrak{I})$, and $\sigma\psi = \pi$. Here π is the natural projection of $\mathfrak{S}$ onto $\mathfrak{R}/x_0\mathfrak{R}$. Both σ and ψ are homogeneous maps of degree zero.

The dth homogeneous piece, $\psi(\mathfrak{I})_d$, of $\psi(\mathfrak{I})$ is obviously spanned by $\psi(F_1), \ldots, \psi(F_n)$. We claim that these elements are linearly independent over k. This argument is the same as the first paragraph of the proof of Theorem 1 in [**3**]. Hence we omit it. Set $V = \psi(\mathfrak{I})_d$. Then $\dim_k V = n$.

Now $\mathfrak{R}$ is a graded ring. We shall denote the jth homogeneous piece of $\mathfrak{R}$ by $\mathfrak{R}_j$. Since $\mathfrak{L}_1, \ldots, \mathfrak{L}_s$ are in generic s-position, we have $\dim_k \mathfrak{R}_j = \min\{s, \nu(j)\}$ for $j = 0, 1 \ldots$. In particular, we have the following dimensions:

$$(4) \qquad \dim_k \mathfrak{R}_j = \begin{cases} \nu(j) & \text{if } j = 0, 1, \ldots, d-1, \\ s & \text{if } j \geqslant d. \end{cases}$$

Since x_0 is not a zero-divisor in $\mathfrak{R}$, multiplication by x_0 is a monomorphism. Thus, equation (4) implies $\mathfrak{R}/x_0\mathfrak{R}$ has the following form:

$$\mathfrak{R}/x_0\mathfrak{R} = \bigoplus_{j=0}^{d} \{\mathfrak{R}_j/x_0\mathfrak{R}_{j-1}\}. \tag{5}$$

In equation (5), the 0th term is k. Now we note that the jth homogeneous piece, $\mathfrak{T}_j$, of $\mathfrak{T}$ has dimension given by $\dim_k \mathfrak{T}_j = \binom{n+j-1}{n-1}$. But $\binom{n+j-1}{n-1} = \dim_k\{\mathfrak{R}_j/x_0\mathfrak{R}_{j-1}\}$ for $j = 0,\dots,d-1$. Thus, σ is an isomorphism in degrees $0,1,\dots,d-1$. Consequently, $\mathfrak{R}/x_0\mathfrak{R} \cong \mathfrak{T}_0 \oplus \cdots \oplus \mathfrak{T}_{d-1} \oplus \{\mathfrak{T}_d/V\}$.

Now let

$$\phi\colon \mathfrak{R}_{d-1}/x_0\mathfrak{R}_{d-2} \to \operatorname{Hom}_k\left(\frac{\mathfrak{R}_1}{x_0\mathfrak{R}_0}, \frac{\mathfrak{R}_d}{x_0\mathfrak{R}_{d-1}}\right)$$

be the natural map induced by multiplication in $\mathfrak{R}/x_0\mathfrak{R}$. Let m denote the homogeneous, maximal ideal in $\mathfrak{R}$. Then the annihilator of $m/x_0\mathfrak{R}$ in $\mathfrak{R}/x_0\mathfrak{R}$ is given by

$$\operatorname{Ann}\{m/x_0\mathfrak{R}\} = \ker\phi \oplus \{\mathfrak{R}_d/x_0\mathfrak{R}_{d-1}\}. \tag{6}$$

Since x_0 is not a zero-divisor in $\mathfrak{R}$, $t(\mathfrak{L}_1,\dots,\mathfrak{L}_s) = \dim_k\{\operatorname{Ann}\{m/x_0\mathfrak{R}\}\}$. Proofs of these last two facts can be found in [5].

Now equation (5) implies that $\{\mathfrak{R}_d/x_0\mathfrak{R}_{d-1}\} \cong \mathfrak{T}_d/V$. Since $\dim_k\{\mathfrak{T}_d/V\} = \binom{n+d-1}{n-1} - n$, the theorem will follow from equation (6) once we have shown that ϕ is a monomorphism. So, we have reduced the proof of the theorem to showing that ϕ is injective when $\mathfrak{L}_1,\dots,\mathfrak{L}_s$ are in uniform position in $\mathbf{A}_k^{n+1}$.

Consider the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{R}_{d-1}/x_0\mathfrak{R}_{d-2} & \xrightarrow{\phi} & \operatorname{Hom}_k\left(\dfrac{\mathfrak{R}_1}{x_0\mathfrak{R}_0}, \dfrac{\mathfrak{R}_d}{x_0\mathfrak{R}_{d-1}}\right) \\ \uparrow \sigma_{d-1} & & \uparrow \sigma_d^* \\ \mathfrak{T}_{d-1} & \xrightarrow{\tilde{\phi}} & \operatorname{Hom}_k(\mathfrak{T}_1, \mathfrak{T}_d) \end{array} \tag{7}$$

In diagram (7), $\tilde{\phi}$ is the obvious monomorphism induced by multiplication in $\mathfrak{T}$. σ_{d-1} is the $(d-1)$-piece of the epimorphism σ. We have previously noted that σ_{d-1} is a k-vector space isomorphism. σ_d^* is the surjection induced by the map $\sigma_d\colon \mathfrak{T}_d \to \mathfrak{R}_d/x_0\mathfrak{R}_{d-1} \cong \mathfrak{T}_d/V$.

Now let $w \in \ker\phi$. Since σ_{d-1} is an isomorphism, there exists a $g \in \mathfrak{T}_{d-1}$ such that $\sigma_{d-1}(g) = w$. The commutativity of diagram (7) implies $\sigma_d^*\tilde{\phi}(g) = 0$. Thus, $g\mathfrak{T}_1 \subseteq V$. In particular, there exist constants $\alpha_{ij} \in k$ such that $gY_i = \sum_{j=1}^{n}\alpha_{ij}\psi(F_j)$, $i = 1,\dots,n$. Now suppose $g \neq 0$. Then $gY_1,\dots,gY_n$ are linearly independent over k. Thus, the matrix (α_{ij}) is invertible. Let (β_{ij}) be the inverse of (α_{ij}). Then $\psi(F_i) = \sum_{j=1}^{n}\beta_{ij}(gY_j)$, $i = 1,\dots,n$. In particular, $\psi(F_1)$ and $\psi(F_2)$ are in the principal ideal (g) in $\mathfrak{T}$. Since $\{X_0, F_1, F_2\}$ is a regular sequence in $\mathfrak{S}$, $\{\psi(F_1), \psi(F_2)\}$ is a regular sequence in $\mathfrak{T}$. But, this last fact implies the grade of (g) is at least two, which is impossible. Therefore, $g = 0$, and ϕ is a monomorphism. This completes the proof of the theorem.

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