UNIFORM $\sigma$-ADDITIVITY IN SPACES OF BOCHNER OR PETTIS INTEGRABLE FUNCTIONS OVER A LOCALLY COMPACT GROUP

NICOLAE DINCULEANU

Abstract. If $G$ is an abelian locally compact group with Haar measure $\mu$, $E$ is a Banach space and $K \subset L^1_G(\mu)$, we give necessary and sufficient conditions for the set \( \{ f \to \int f \, d\mu : f \in K \} \) to be uniformly $\sigma$-additive in terms of uniform convergence on $K$, for the topology $\sigma(L^1_E, L^\infty_E)$ of convolution and translation operators. In case $E = R$, this gives a new characterization of relatively weakly compact sets $K \subset L^1$.

1. Introduction. In this paper we consider the space $L^1_E$ of Bochner integrable functions and the space $\mathcal{L}^1_E$ of Pettis integrable functions over an abelian locally compact group $G$ endowed with a Haar measure $\mu$, and we give a characterization of uniform $\sigma$-additivity in terms of uniform convergence—in the topology $\sigma' = \sigma(L^1_E, L^\infty_E)$, respectively in the weak topology of $\mathcal{L}^1_E$—of convolution operators and translation operators. If $E = R$, this yields a new characterization of relative weak compactness in $L^1$.

The convolution of Bochner integrable functions has been studied in [2] and has been extended in [4] for Pettis integrable functions.

Similar results have been obtained in a previous paper [3], where we give a characterization of uniform $\sigma$-additivity in the spaces $L^1_E$ and $\mathcal{L}^1_E$ over a measure space $(X, \Sigma, \mu)$, in terms of uniform convergence, in the $\sigma'$-topology or in the weak topology, of conditional expectations.

2. Uniform $\sigma$-additivity in the Lebesgue space $L^1_E$. Let $G$ be an abelian locally compact additive group endowed with a Haar measure $\mu$; let $E$ be a Banach space and $L^1_E$ be the space of Bochner $\mu$-integrable functions $f : G \to E$. For each relatively compact neighborhood $V$ of 0 in $G$, we choose a function $u_V$ on $G$ which is positive, bounded, symmetric (i.e. $u_V(-t) = u_V(t)$), vanishes outside $V$ and $\int u_V \, d\mu = 1$. If $\mathcal{V}$ is a base of relatively compact neighborhoods of 0 in $G$, we call $(u_V)_{V \in \mathcal{V}}$ an approximate unit. We denote by $u_V \ast f$ the convolution: $u_V \ast f(t) = \int u_V(t - s)f(s) \, ds$, for $t \in G$. For $h \in G$ we denote by $T^h$ the translation operator, defined by $(T^hf)(t) = f(t + h)$, for $t \in G$. If $f \in L^1_E$, we denote by $f_\mu$ the measure defined for any Borel set $A \subset G$ by $(f_\mu)(A) = \int_A f \, d\mu$. Finally, we denote by $\sigma'$ the topology $\sigma(L^1_E, L^\infty_E)$ on

Received by the editors September 15, 1981.
1980 Mathematics Subject Classification. Primary 28B05; Secondary 43A20, 46E40, 46G10.
Key words and phrases. Locally compact group, Haar measure, Bochner integral, Pettis integral, convolution, translation, approximate unit, uniform $\sigma$-additivity, uniform convergence.

©1983 American Mathematical Society
0002-9939/82/0000-1053/$02.50

627
If \( E \) has the Radon-Nikodym property, then the \( \sigma' \)-topology is the weak topology of \( L^1_E \).

**Theorem 1.** Let \( K \subset L^1_E \) be a set.

I. The set \( |K| \mu = \{ |f| \mu; f \in K \} \) is uniformly \( \sigma \)-additive, if and only if the following conditions are satisfied:

(a) \( K \) is bounded in \( L^1_E \);

(b) For every countable subset \( K_0 \subset K \), there exists a decreasing sequence \( (V_n) \) of neighborhoods of 0 in \( G \), such that either

\[
(b_1) \lim_{n} u_{V_n} \ast f = f \text{ in } L^1_E \text{ for the } \sigma \text{-topology, uniformly for } f \in K_0; \text{ or}
\]

\[
(b_2) \lim_{h \to 0} T^h f = f \text{ in } L^1_E \text{ for the } \sigma' \text{-topology, uniformly for } f \in K_0;
\]

(c) \( \lim_{C} x_C f = f \), strongly in \( L^1_E \), uniformly for \( f \in K \), where the limit is taken along the increasing net of all compact subsets of \( G \).

(Condition (c) is superfluous if all functions of \( K \) vanish outside a common compact set; in particular, if \( G \) is compact.)

II. If \( |K| \mu \) is uniformly \( \sigma \)-additive, then

\[
(b_1') \lim_{V} u_{V} \ast f = f \text{ and}
\]

\[
(b_2') \lim_{h \to 0} T^h f = f,
\]

in \( L^1_E \), for the \( \sigma' \)-topology, uniformly for \( f \in K \).

**Proof.** Assume first conditions (a), (b), and (c) satisfied and prove that \( |K| \mu \) is uniformly \( \sigma \)-additive. Let \( C \subset G \) be a compact set and \( \phi \in L^1 \cap L^\infty \) a function with compact support \( V \).

(A) The set \( \phi \ast (x_C K) = \{ \phi \ast (x_C f); f \in K \} \) is bounded in \( L^1_E \). In fact, if \( f \in K \), then

\[
\| \phi \ast (x_C f) \|_1 \leq \| \phi \|_1 \| x_C f \|_1 \leq M \| \phi \|_1,
\]

where \( M = \sup \{ \| f \|_1; f \in K \} \).

(B) The set \( \phi \ast (x_C K) \mu \) is uniformly \( \sigma \)-additive. In fact the set \( \phi \ast (x_C K) \) is bounded in \( L^\infty_E \):

\[
\| \phi \ast (x_C f) \|_\infty \leq \| \phi \|_\infty \| x_C f \|_1 \leq M \| \phi \|_1, \text{ for } f \in K.
\]

It follows that the set \( \phi \ast (x_C K) \mu \) is uniformly \( \sigma \)-continuous. Since all the functions of \( \phi \ast (x_C K) \) vanish outside the compact set \( C + V \), the set \( \phi \ast (x_C K) \mu \) is uniformly \( \sigma \)-additive.

(C) For every \( g \in L^\infty_E \) and \( f \in K \) we have

\[
\left| \int \langle \phi \ast (x_C f) - f, g \rangle \, d\mu \right| \leq \left| \int \langle \phi \ast (x_C f - f), g \rangle \, d\mu \right| + \left| \int \langle \phi \ast f - f, g \rangle \, d\mu \right|
\]

\[
\leq \| \phi \|_1 \| x_C f - f \|_1 \| g \|_\infty + \left| \int \langle \phi \ast f - f, g \rangle \, d\mu \right|.
\]

From condition (c) we deduce that there is an increasing sequence \( (C_n) \) of compact sets such that \( \lim_n \| x_{C_n} f - f \|_1 = 0 \), uniformly for \( f \in K \). Let \( K_0 \subset K \) be a countable set. Taking above \( C = C_n \) and \( \phi = u_{V_n} \), where \( (V_n) \) is the sequence stated in condition (b), we deduce that

\[
\lim_n u_{V_n} \ast (x_{C_n} f) = f, \text{ in } L^1_E
\]
for the $\sigma'$-topology, uniformly for $f \in K_0$. Since, for each $n$, the set $|u_{r^n} \ast (\chi_{c^n} K_0)| \mu$ is bounded and uniformly $\sigma$-additive, from Lemma 1b in [3] we deduce that $|K_0| \mu$ is also uniformly $\sigma$-additive. Since $K_0$ was an arbitrary countable set in $K$, it follows that $|K| \mu$ is uniformly $\sigma$-additive. If conditions (a), (b2) and (c) are satisfied, then $|K| \mu$ is again uniformly $\sigma$-additive. since by Proposition 12 in [4], condition (b2) implies (b1). We remark that in [4], the implication $b_2 \Rightarrow b_1$ is stated for the weak topology, but the same proof is valid for the $\sigma'$-topology.

Conversely, assume $|K| \mu$ is uniformly $\sigma$-additive.

(D) $K$ is bounded in $L^1$. In fact, we can find a compact set $B \subset G$ such that $\int_{G-B} |f| \, d\mu \leq 1$ for all $f \in K$. Since $|K| \mu$ is uniformly absolutely $\mu$-continuous, there is $\eta > 0$ such that if $\mu(A) \leq \eta$, then $\int |f| \, d\mu \leq 1$ for all $f \in K$. Since the Haar measure is diffuse, it has the Darboux property: there is a finite family of disjoint Borel sets $A_1, \ldots, A_n$ with union $B$, such that $\mu(A_i) \leq \eta$ for $i = 1, \ldots, n$. It follows that $\int_B |f| \, d\mu \leq n$ for all $f \in K$, hence $\int |f| \, d\mu \leq n + 1$ for all $f \in K$; consequently $K$ is bounded.

(E) Since $|K| \mu$ is uniformly $\sigma$-additive, for every $\varepsilon > 0$ there is a compact set $C \subset G$ such that

$$\int_{G-C} |f| \, d\mu < \varepsilon, \quad \text{for all } f \in K,$$

that is

$$\int |\chi_C f - f| \, d\mu < \varepsilon, \quad \text{for all } f \in K$$

and condition (c) follows.

(F) Let $f \in L^1, g \in L^\infty, \lambda > 0, h \in G$, and $C$ be an integrable subset. Then

$$\left| \int \langle T^h f - f, g \rangle \, d\mu \right| \leq 2\|g\|_\infty \int_{G-C} |f| \, d\mu + 2\|g\|_\infty \int_{\{f \sim \lambda\}} |f| \, d\mu$$

$$+ \lambda \|T^{-h} (\chi_C g) - \chi_C g\|_1 + \lambda \|g\|_\infty \|\chi_C - \chi_{C-h}\|_1.$$ 

In fact

$$\left| \int \langle T^h f - f, g \rangle \, d\mu \right| \leq \left| \int \langle T^h (f \chi_C) - f \chi_C, g \rangle \, d\mu \right|$$

$$+ \left| \int \langle T^h (f \chi_C) - f \chi_C, g \rangle \, d\mu \right|.$$ 

The first term can be written

$$\left| \int \langle T^h (f \chi_C - f \chi_C, g) \rangle \, d\mu \right| = \left| \int \langle f \chi_C, T^{-h} g - g \rangle \, d\mu \right|$$

$$\leq 2\|g\|_\infty \int_{G-C} |f| \, d\mu.$$
For the second term we have \( T^h(f \chi_C) = \chi_{C-h} T^h(f \chi_C) \), hence
\[
\left| \int \langle T^h(f \chi_C) - f \chi_C, g \rangle \, d\mu \right| = \left| \int \langle f \chi_C, T^{-h}(g \chi_{C-h}) - g \chi_C \rangle \, d\mu \right|
\]
\[
\leq \left| \int_{C \cap \{|y| > \lambda\}} \langle f, T^{-h}(g \chi_{C-h}) - g \chi_C \rangle \, d\mu \right|
+ \left| \int_{C \cap \{|y| < \lambda\}} \langle f, T^{-h}(g \chi_{C-h}) - g \chi_C \rangle \, d\mu \right|
\]
\[
\leq 2\|g\|_\infty \int_{\{|y| > \lambda\}} |f| \, d\mu + \lambda \|T^{-h}(g \chi_{C-h}) - g \chi_C\|_1
\]
\[
\leq 2\|g\|_\infty \int_{\{|y| > \lambda\}} |f| \, d\mu + \lambda \|T^{-h}(g \chi_C) - g \chi_C\|_1
\]
\[
+ \lambda \|T^{-h}g(\chi_{C-h} - \chi_C)\|_1,
\]
and this last term is smaller than \( \lambda \|g\|_\infty \|\chi_{C-h} - \chi_C\|_1 \).

(G) We can now prove conditions (b2') and (b1'). Let \( g \in L^\infty, g \neq 0 \) and \( \epsilon > 0 \).
Take \( C \subset G \) such that
\[
\int_{G-C} |f| \, d\mu < \epsilon/(8\|g\|_\infty), \quad \text{for all } f \in K.
\]
Take also \( \lambda > 0 \) such that
\[
\int_{\{|y| > \lambda\}} |f| \, d\mu < \epsilon/(8\|g\|_\infty), \quad \text{for all } f \in K.
\]
We can find a symmetric neighborhood \( V \) of 0 such that for all \( h \in V \) we have
\[
\|T^{-h}(\chi_C g) - \chi_C g\|_1 < \epsilon/4,
\]
and
\[
\|\chi_C - \chi_{C-h}\|_1 = \|\chi_C - T^h \chi_C\|_1 < \epsilon/(4\lambda \|g\|_\infty).
\]
Then, for \( h \in V \) and all \( f \in K \) we have, from step (F),
\[
\left| \int \langle T^h f, g \rangle \, d\mu \right| < \epsilon;
\]
that is \( \lim_{h \to 0} T^h f = f \), in \( L^1_\epsilon \) for the \( \sigma' \)-topology, uniformly for \( f \in K \). This proves condition (b2'); and condition (b1') follows from Proposition 12 in [3].

(H) To prove condition (b2), let \( K_0 \subset K \) be a countable subset. The proof of condition (b1) is the same as in step (G).

Let \( R_0 \) be a countable ring of relatively compact Borel subsets of \( G \), such that any function of \( K_0 \) is the limit \( \mu \)-a.e. and in \( L^1_\epsilon \) of step functions over \( R_0 \).

Since for each \( f \in L^1_\epsilon \) we have \( \lim_{h \to 0} T^h f = f \), strongly in \( L^1_\epsilon \), we can find a decreasing sequence \( (V_n) \) of symmetric neighborhoods of 0, such that \( \lim_{h \in V_{n, n} \to 0} T^h \chi_A = \chi_A \), strongly in \( L^1 \) for every \( A \in R_0 \). Next, we choose arbitrarily a sequence \( (h_n) \) such that \( h_{2n-1} = -h_{2n} \in V_n \) for every \( n \). Then \( \lim_n T^{h_n} \chi_A = \chi_A \) strongly in \( L^1 \) for every \( A \in R_0 \). Consider the group \( \Gamma \subset G \) generated by the
sequence \((h_n)\). Then the set \(L\) of linear combinations of functions of the form \((T^{a_1}X_{A_1})(T^{a_2}X_{A_2})\cdots(T^{a_k}X_{A_k})\) with \(a_1,\ldots,a_k \in \Gamma\) and \(A_1,\ldots,A_k \in R_0\), is an algebra of \(\mu\)-integrable functions, invariant with respect to \(T^a\) for any \(a \in \Gamma\). Moreover, \(\lim_n T^{h_n}X = X\), in \(L^1\), for all \(X \in L\).

It is enough to check this for the functions of the form \(X = T^{a_1}X_{A_1}(T^{a_2}X_{A_2})\). We have \(\lim_n T^{h_n}X = X\), \(\mu\)-a.e. and since \(|T^{h_n}X| \leq X_{A_1} + (A_1 \cup A_2)\), we can apply Lebesgue's dominated convergence theorem and deduce that \(\lim_n T^{h_n}X = X\) in \(L^1\).

Moreover, this last equality remains valid for \(X\) in the closure of \(L\) in \(L^1\), since \(\sup_n \|T^{h_n}\| = 1\).

The class \(\Lambda = \{A; X_A \in L\}\) is a ring containing \(R_0\), and the class \(\{X_A; A \in \Lambda\}\) is invariant with respect to \(T^a\) for all \(a \in \Gamma\). All functions of \(L\) vanish \(\mu\)-a.e. outside a \(\sigma\)-finite set \(X_0\).

The \(\delta\)-ring \(\Sigma_0\) generated by \(\Lambda\) is the completion of \(\Lambda\) for the semidistance \(\rho(A, B) = \|X_A - X_B\|_1\), and can be obtained—modulo negligible sets—as closure in \(L^1\) of the set of functions \(X_A\) with \(A \in \Lambda\). It follows that the class \(\{X_A; A \in \Sigma_0\}\) is invariant with respect to \(T^a\) for all \(a \in \Gamma\), since the class \(\{X_A; A \in \Lambda\}\) has this property, and since this property is preserved by passing to limits in \(L^1\).

We deduce then that for any Banach space \(F\), the space \(L^1_F(X_0, \Sigma_0, \mu)\) is invariant with respect to \(T^a\) for all \(a \in \Gamma\), and that for every \(f \in L^1_F(X_0, \mu)\) we have \(\lim_n T^{h_n}f = f\), strongly in \(L^1_F\).

In fact this property is valid for all step functions, and \(\sup_n \|T^{h_n}\| = 1\).

We are now ready to prove condition (b). Let \(g \in L^\infty_E, g \neq 0\), and \(\varepsilon > 0\). The conditional expectation \(g' = E(g \mid \Sigma_0)\) is defined since the space \((G, \mu)\) is localizable (see [1]).

We can consider \(g' \in L^\infty_E(X_0, \Sigma_0, \mu)\). Since \(K_0 \subset L^1_E(X_0, \Sigma_0, \mu)\) and \(|K_0| \mu\) is uniformly \(\sigma\)-additive, there is a set \(C \in \Sigma_0\) such that

\[\int_{C-C} |f| \, d\mu < \varepsilon / (8\|g\|_\infty), \quad \text{for all } f \in K_0.\]

Also let \(\lambda\) be such that

\[\int_{|f| > \lambda} |f| \, d\mu < \varepsilon / (8\|g\|_\infty), \quad \text{for all } f \in K_0.\]

Since \(X_Cg' \in L^1_E(X_0, \Sigma_0, \mu)\), we have \(\lim_n T^{h_n}(X_Cg') = X_Cg'\), strongly in \(L^1_E\); and we have also

\[\lim_n \|X_C - X_{C-h_n}\|_1 = \lim_n \|X_C - T^{h_n}X_C\|_1 = 0.\]

Let \(n_\varepsilon\) be such that for \(n \geq n_\varepsilon\) we have

\[\|T^{h_n}(X_Cg') - X_Cg'\|_1 < \varepsilon / (4\lambda)\]

and

\[\|X_C - X_{C-h_n}\|_1 < \varepsilon / (4\lambda\|g\|_\infty)\]
Then, for any $f \in K_0$ and any $n \geq n_*$, we deduce from step (F),
\[ \left| \int \langle T^h f, g \rangle \, d\mu \right| = \int \langle T^h f, g' \rangle \, d\mu \leq \varepsilon, \]
that is $\lim_n T^h f = f$, in $L^1_E$ for the $\sigma'$-topology, uniformly for $f \in K_0$. Since the sequence $(h_n)$ was arbitrary, it follows that
\[ \lim_{n \to \infty} T^h f = f, \quad \text{in } L^1_E, \]
for the $\sigma'$-topology, uniformly for $f \in K$ and so, condition (b2) is proved. Condition (b1) then follows from Proposition 12 in [4]; and this completes the proof of the theorem.

**Remark.** The $\sigma'$-topology cannot be replaced by the weak topology. There are examples (to be published in a joint paper with Jürgen Batt) of relatively weakly compact sets $K \subset L^1_E$ over the circle group, such that the limits in (b1) and (b2) for the weak topology are false.

3. **Uniform $\sigma$-additivity in the Pettis space $L^1_E$.** We denote by $\mathcal{L}^1_E$ the Pettis space of functions $f : G \to E$ which are strongly $\mu$-measurable and Pettis integrable, endowed with the Pettis norm
\[ (f)_1 = \sup \left\{ \int |\langle f, x' \rangle| \, d\mu ; x' \in E_1' \right\} \]
where $E_1'$ is the unit ball of $E'$. A set $F \subset E_1'$ is norming for a set $K \subset \mathcal{L}^1_E$, if
\[ |f(t)| = \sup \{ |\langle f(t), x' \rangle| ; x' \in F \}, \mu\text{-a.e. for every } f \in K. \]

**Theorem 2.** Let $K \subset \mathcal{L}^1_E$ be a set.

1. The set $K\mu$ is uniformly $\sigma$-additive, if and only if the following conditions are satisfied:
   (a) $K$ is bounded in $\mathcal{L}^1_E$;
   (b) For every countable subset $K_0 \subset K$ there is a decreasing sequence $(V_n)$ of neighborhoods of 0 and a countable subset $E_0' \subset E_1'$, norming for $K_0$, such that either
      (b1) $\lim_n \langle u_{V_n} \ast f, x' \rangle = \langle f, x' \rangle$, weakly in $L^1$, uniformly for $f \in K_0$ and $x' \in E_0'$;
      or
      (b2) $\lim_{n \to \infty} \langle T^h f, x' \rangle = \langle f, x' \rangle$, weakly in $L^1$, uniformly for $f \in K_0$ and $x' \in E_0'$;
   (c) $\lim_C f|C \mu = f$, strongly in $\mathcal{L}^1_E$, uniformly for $f \in K$, the limit being taken along the increasing net of all compact subsets of $G$.

II. If $K\mu$ is uniformly $\sigma$-additive, then:
   (b1') $\lim_V \langle u_V \ast f, x' \rangle = \langle f, x' \rangle$ and
   (b2') $\lim_{h \to 0} \langle T^h f, x' \rangle = \langle f, x' \rangle$
weakly in $L^1$, uniformly for $f \in K$ and $x' \in E_1'$.

**Proof.** Assume first conditions (a), (b) and (c) satisfied. Let $K_0 \subset K$ be a countable set, and $E_0'$ the set corresponding to $K_0$ by condition (b) above.

Then the set $\langle K_0, E_0' \rangle = \{ (f, x') ; f \in K_0, x' \in E_0' \}$ is countable and satisfies conditions (a), (b) and (c) of Theorem 1, in the space $L^1$. It follows that $\langle K_0, E_0' \rangle \mu$ is
uniformly $\sigma$-additive; and then $K_0\mu$ is also uniformly $\sigma$-additive; therefore $K\mu$ is uniformly $\sigma$-additive.

Conversely, assume $K\mu$ is uniformly $\sigma$-additive; let $K_0 \subset K$ be countable, and let $E_0' \subset E_1'$ be a countable set, norming for $K_0$. The set $\langle K_0, E_0' \rangle$ is countable and uniformly $\sigma$-additive. From Theorem 1 we deduce:

(a) The set $\langle K_0, E_0' \rangle$ is bounded in $L_2$; hence $K_0$ is bounded in $L_2$; therefore $K$ is bounded in $L_2$.

(b) There exists a decreasing sequence $(V_n)$ of neighborhoods of 0, satisfying conditions $(b_1)$ and $(b_2)$ of this theorem;

(c) $\lim_n \langle \chi f, x' \rangle = \langle f, x' \rangle$, strongly in $L_2$ uniformly for $f \in K$ and $x' \in E_0'$, which is equivalent to condition (c) of this theorem.

Finally, to obtain $(b_1')$ and $(b_2')$ we apply the second part of Theorem 1 to the set $\langle K, E_1' \rangle$.

Note. We take this opportunity to mention that Theorems 2(iii) and 4(iii) in [3] are valid without the condition $\sup\{|f(t)| : t \in K\} < \infty$, $\mu$-a.e. The proof will be given in a forthcoming paper, for a more general situation.

References


Department of Mathematics, University of Florida, Gainesville, Florida 32611