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F_p CLASSES AND HYPERGEOMETRIC SERIES

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ABSTRACT. For $\mathrm{Re}(p) \geq 0$, let $F_p = \{f: f(z) = \int(1 - xz)^{-p}\, d\mu(x),\ |z| < 1,\ \mu$ a probability measure on $|x| = 1\}$, and let $F_p \cdot F_q = \{f \cdot g: f$ in F_p, g in $F_q\}$. Brickman, Hallenbeck, MacGregor and Wilken proved that $p > 0$ and $q > 0$, then $F_p \cdot F_p \subseteq F_{p+q}$. Kent Pearce recently proved a converse result: if $F_p \cdot F_q \subseteq F_{p+q}$, then $p > 0$ and $q > 0$, or $p = q = 1 + it$ for some real t. The case $p = q = 1 + it$, $t \neq 0$, will be excluded. Consequently a full converse of the above theorem holds.

Introduction. Let $F_p = \{f: f(z) = \int_{|x|=1}(1 - xz)^{-p}\, d\mu(x),\ |z| < 1,\ \mu$ a probability measure on $|x| = 1\}$, where $\mathrm{Re}\, p > 0$. Also let $F_p \cdot F_q = \{f \cdot g: f \in F_p,\ g \in F_q\}$, where $\cdot$ denotes the Hadamard convolution. Pearce [1] proved that if $F_p \cdot F_q \subseteq F_{p+q}$ then $p > 0$ and $q > 0$, or $p = q = 1 + it$, t real. In this paper, the case $p = q = 1 + it$, $t \neq 0$, will be excluded, using an identity involving hypergeometric functions.

Pierce [1] used the following theorem in establishing his result.

THEOREM A. *If $F_p \cdot F_q \subseteq F_{p+q}$ then*

$$\mathrm{Re}\left(\frac{1}{1 - xz} - {}_2F_1\left(\begin{matrix} 1, q \\ p + q \end{matrix}; \frac{(y - x)z}{1 - xz}\right)\right) > \frac{1}{2} \tag{1}$$

for all x, y and z such that $|x| = |y| = 1$ and $|z| < 1$.

Here the hypergeometric function ${}_2F_1(\begin{smallmatrix} a, b \\ c \end{smallmatrix}; z)$ is defined for $|z| < 1$ by

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k, \tag{2}$$

where $(a)_n = a(a + 1)(a + 2) \cdots (a + n - 1)$, $n > 0$, and $(a)_0 = 1$.

If $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, then the ${}_2F_1$ has an integral representation

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \frac{\Gamma(c)}{\Gamma(c - b)\Gamma(b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - tz)^{-a}\, dt. \tag{3}$$

This provides an analytic continuation of the ${}_2F_1$ to $\mathbf{C} - [1, \infty)$ under the restriction $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$. Using some recurrence relations involving the parameters, the ${}_2F_1$ can be analytically continued to $\mathbf{C} - [1, \infty)$ for all a, b, and c such that $c \neq 0, -1, \ldots$ (see Slater [3]).

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Also we note the simple identity

$$(4)\qquad {}_2F_1\left(\begin{matrix} a, b \\ a \end{matrix}; z\right) = (1-z)^{-b}, \qquad a \neq 0, -1, \ldots.$$

The main result.

THEOREM 1. *If $F_{1+it} \cdot F_{1+it} \subseteq F_{2+2it}$, t real, then $t = 0$.*

PROOF. By Theorem A we must have

$$(5)\quad \operatorname{Re}\left(\frac{1}{1-xz}\,{}_2F_1\left(\begin{matrix} 1, 1+it \\ 2+2it \end{matrix}; \frac{(y-x)z}{1-xz}\right)\right) > \frac{1}{2}, \qquad |x| = |y| = 1, |z| < 1.$$

We will assume $t \neq 0$ and obtain a contradiction. First we state a lemma proved by Slater [**3**, p. 34].

$$(6)\quad {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,{}_2F_1\left(\begin{matrix} a, b \\ a+b-c+1 \end{matrix}; 1-z\right)$$
$$+ (1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\,{}_2F_1\left(\begin{matrix} c-a, c-b \\ c-a-b+1 \end{matrix}; 1-z\right),$$

$c \neq 0, -1, -2, \ldots$, $a+b-c \neq 0, \pm 1, \pm 2, \ldots$, $|\operatorname{Arg}(z)| < \pi$, $|\operatorname{Arg}(1-z)| < \pi$.

The lemma is useful in that it reveals the behavior of a ${}_2F_1$ when $z \to 1$ with rather relaxed restrictions on the parameters a, b and c. It is clear from (6) that if $\operatorname{Re}(c-a-b) > 0$ then

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) \to \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{as } z \to 1.$$

If $\operatorname{Re}(c-a-b) < 0$ then the ${}_2F_1\left(\begin{smallmatrix} a, b \\ c \end{smallmatrix}; z\right)$ goes to infinity like $(1-z)^{c-a-b}$ as $z \to 1$. The case where $\operatorname{Re}(c-a-b) = 0$ can also be treated by (6) and that is what will make the proof of this theorem work.

Now let $z = r\bar{y}$, $0 < r < 1$, and $x\bar{y} = \zeta \neq 1$. Then by (6),

(7)

$${}_2F_1\left(\begin{matrix} 1, 1+it \\ 2+2it \end{matrix}; \frac{(y-x)z}{1-xz}\right) = {}_2F_1\left(\begin{matrix} 1, 1+it \\ 2+2it \end{matrix}; \frac{r(1-\zeta)}{1-\zeta r}\right)$$
$$= \frac{\Gamma(2+2it)\Gamma(it)}{\Gamma(1+2it)\Gamma(1+it)}\,{}_2F_1\left(\begin{matrix} 1, 1+it \\ 1-it \end{matrix}; \frac{1-r}{1-\zeta r}\right)$$
$$+ \left(\frac{1-r}{1-\zeta}\right)^{it}\frac{\Gamma(2+2it)\Gamma(-it)}{\Gamma(1)\Gamma(1+it)}\,{}_2F_1\left(\begin{matrix} 1+2it, 1+it \\ 1+it \end{matrix}; \frac{1-r}{1-\zeta r}\right).$$

Here the powers are computed using the principal value of the arguments, for this was assumed in (6). In order to simplify (7) we use (4) and the following identities [**2**, pp. 12–24].

$$(8)\quad \begin{aligned} &\text{(i)} && \Gamma(z+1) = z\Gamma(z), && z = 0, -1, -2, \ldots, \\ &\text{(ii)} && \sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z), && z \neq 0, -\tfrac{1}{2}, -1, -\tfrac{3}{2}, \ldots, \\ &\text{(iii)} && \Gamma(1/2) = \sqrt{\pi}. \end{aligned}$$

We then obtain

$$(9)\quad \frac{1}{1-xz}\,{}_2F_1\left(\begin{matrix}1, 1+it\\ 2+2it\end{matrix}; \frac{(y-x)z}{1-xz}\right)$$
$$= \frac{2-i/t}{1-\zeta r}\,{}_2F_1\left(\begin{matrix}1, 1+it\\ 1-it\end{matrix}; \frac{1-r}{1-\zeta r}\right)$$
$$+ (1-r)^{it}\Gamma\left(\frac{3}{2}+it\right)\Gamma(-it)\left(\frac{1}{2}(1-\zeta)r\right)^{-1-2it}(1-\zeta r)^{it},$$

$\zeta = x\bar{y}$, $z = r\bar{y}$, $0 < r < 1$. The right side of (9) has the form

$$(10)\qquad A(r,\zeta) + (1-r)^{it}B(r,\zeta),$$

where

$$\lim_{r\to 1^-} A(r,\zeta) = \frac{2-i/t}{1-\zeta}$$

and

$$(11)\qquad \lim_{r\to 1^-} B(r,\zeta) = \left(\frac{1}{2}(1-\zeta)\right)^{-1-2it}(1-\zeta)^{it}\Gamma\left(\frac{3}{2}+it\right)\Gamma(-it).$$

Note that as $r \to 1^-$, $(1-r)^{it} = \exp(it\,\mathrm{Ln}(1-r))$ is a complex number which revolves around the origin more and more rapidly while the modulus is one. Consequently, we must have

$$(12)\qquad \mathrm{Re}\,A(r,\zeta) > \tfrac{1}{2}, \qquad 0 < r < 1, |\zeta| = 1, \zeta \neq 1.$$

Letting $r \to 1^-$, we have

$$(13)\qquad \mathrm{Re}\left(\frac{2-i/t}{1-\zeta}\right) > \frac{1}{2}.$$

But if we let $\zeta = e^{i\theta}$

$$\mathrm{Re}\left(\frac{2-i/t}{1-\zeta}\right) = 1 - \frac{(1+\cos\theta)}{2t\sin\theta}$$

and it is clear that a nonzero value of θ can be chosen so that the above expression is less than $\frac{1}{2}$. We have obtained a contradiction and the theorem is proved. □

An obvious consequence of this theorem is

COROLLARY. *If $F_p \cdot F_q \subseteq F_{p+q}$ then $p > 0$ and $q > 0$.*

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