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INSERTING A_p-WEIGHTS

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ABSTRACT. Necessary and sufficient conditions are given for inserting a single weight $w \in A_p$ between u and v, i.e., $c_1 u \leq w \leq c_2 v$.

1. Let (u, v) be a pair of nonnegative measurable functions on $\mathbf{R}^n$. We ask under what conditions is it possible to find a function w with $c_1 u \leq w \leq c_2 v$ so that both pairs (u, w), (w, v) are in A_p, or even better w itself is in A_p. We recall that $w \in A_p$ iff $(\int_Q w)(\int_Q w^{1-p'})^{p-1} \leq C|Q|^p$, and $(u, v) \in A_p$ iff $(\int_Q u)(\int_Q v^{1-p'})^{p-1} \leq C|Q|^p$. These classes were introduced by Muckenhoupt [5] and play an important role in weighted norm inequalities for many operators, as the Hardy-Littlewood maximal operator $Mf(x) = \sup_{x \in Q} |Q|^{-1} \int_Q |f|\, dt$, the Hilbert transform, and many others.

The single weight problem is easier to handle than the double weight problem, and thus the weighted norm inequalities for (u, v) would follow from those of $w \in A_p$ if $c_1 u \leq w \leq c_2 v$.

2. Throughout $1 < p < \infty$, unless otherwise noted. We need the following lemmas.

LEMMA 1. *$(u, v) \in A_p$ implies $(v^{1-p'}, u^{1-p'}) \in A_{p'}$, $1/p + 1/p' = 1$.*

PROOF. This is simply $(1 - p')(1 - p) = 1$.

LEMMA 2. *Let $(u, v) \in A_p$ and let $0 < \delta < 1$. If $(q-1)/(p-1) = \delta$, then $(u^\delta, v^\delta) \in A_q$.*

PROOF. Note that $1 < q < p$. Since $(v^{1-p'}, u^{1-p'}) \in A_{p'}$ and $q' > p'$, we have $(v^{1-p'}, u^{1-p'}) \in A_{q'}$. We use Lemma 1 again and infer that $(u^{(1-p')(1-q)}, v^{(1-p')(1-q)}) \in A_q$, that is $(u^\delta, v^\delta) \in A_q$.

COROLLARY. *If $(u, v) \in A_p$, $0 < \delta < 1$, then $\|Mf\|_{p,u^\delta} \leq A_\delta \|f\|_{p,v^\delta}$.*

PROOF. Since $(u^\delta, v^\delta) \in A_q$ and $p > \delta(p-1) + 1 = q$, we get $u^\delta(Mf > y) \leq c\|f\|^q_{q,v^\delta}/y^q$ [5], and thus $\|Mf\|_{p,u^\delta} \leq A_\delta \|f\|_{p,v^\delta}$ by the Marcinkiewicz interpolation theorem.

3. The following theorem is the key to our problem. Its proof is a two weight adaptation of the Coifman, Jones and Rubio de Francia argument of factoring an A_p-weight [1].

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THEOREM 1. *Assume that*

(i) $\|Mf\|_{p,u} \leqslant A\|f\|_{p,v}$,

(ii) $\|Mf\|_{p',v^{1-p'}} \leqslant B\|f\|_{p',u^{1-p'}}$.

Then there are functions $w_j \geqslant 0$ *such that* $u^{1/p}Mw_j \leqslant C_j w_j v^{1/p}$, $j = 1, 2$, *and* $u^{1/p}v^{1/p'} = w_1 w_2^{1-p}$.

PROOF. We may assume that $p \geqslant 2$ (if $1 < p < 2$ simply interchange below the role of (i) and (ii)). Let $s = p/p'$, and $M_s f = M(f^s)^{1/s}$. The operator $S(f) = u^{1/p}M(fv^{-1/p}) + v^{-1/ps}M_s(fu^{1/ps})$ satisfies $\|S(f)\|_p \leqslant C\|f\|_p$. We let $\bar{C} > C$, and for $u_0 \in L^p$, $u_0 > 0$, let $U = \sum_0^\infty S^j(u_0)/\bar{C}^j$. Then $U \in L^p$ and $S(U) \leqslant \bar{C}U$. Hence

$$M(Uv^{-1/p})u^{1/p} \leqslant S(U) \leqslant \bar{C}U \quad \text{and} \quad v^{-1/ps}M_s(Uu^{1/ps}) \leqslant \bar{C}U.$$

Thus, if $w_2 = Uv^{-1/p}$, we get $u^{1/p}Mw_2 \leqslant c_2 w_2 v^{1/p}$, and if we let $w_1 = U^s u^{1/p}$, we get $u^{1/p}Mw_1 \leqslant c_1 w_1 v^{1/p}$, and $u^{1/p}v^{1/p'} = w_1 w_2^{1-p}$.

THEOREM 2. *Let* $(u, v) \in A_p$ *and* $0 < \delta < 1$. *Then there exists* $w = w_\delta \in A_p$ *such that* $c_1 u^\delta \leqslant w \leqslant c_2 v^\delta$.

PROOF. Choose $0 < \varepsilon, \eta < 1$ such that $\delta = \varepsilon\eta$. From the corollary we know that $\|Mf\|_{pu^\varepsilon} \leqslant A\|f\|_{pv^\varepsilon}$, $\|Mf\|_{p',v^{\varepsilon(1-p')}} \leqslant B\|f\|_{p',u^{\varepsilon(1-p')}}$. Thus by Theorem 1, $u^{\varepsilon/p}v^{\varepsilon/p'} = w_1 w_2^{1-p}$, where $Mw_j \leqslant c_j w_j (v/u)^{\varepsilon/p}$. Note that

$$v^\varepsilon = w_1\left(\frac{v}{u}\right)^{\varepsilon/p} w_2^{1-p} \geqslant cMw_1(Mw_2)^{1-p} \geqslant cw_1 \cdot w_2^{1-p}\left(\frac{v}{u}\right)^{\varepsilon(1-p)/p} = cu^\varepsilon,$$

and thus $c_1 u^\delta \leqslant (Mw_1)^\eta (Mw_2)^{\eta(1-p)} \leqslant c_2 v^\delta$. Since $0 < \eta < 1$, $(Mw_j)^\eta \in A_1$, a result due to Coifman. It is easy to see that $w = (Mw_1)^\eta (Mw_2)^{\eta(1-p)} \in A_p$.

We are now able to give necessary and sufficient conditions for inserting A_p-weights.

THEOREM 3. *Let* (u, v) *be a pair of nonnegative functions. Then there exists* $w \in A_p$ *with* $c_1 u \leqslant w \leqslant c_2 v$ *if and only if there is* $\tau > 1$ *such that* $(u^\tau, v^\tau) \in A_p$.

PROOF. Since w and $w^{1-p'}$ obey a reverse Hölder inequality **[5]**, there is $\tau > 1$ such that $w^\tau \in A_p$, and thus $(u^\tau, v^\tau) \in A_p$. Conversely, if $(u^\tau, v^\tau) \in A_p$, then Theorem 2 with $\delta = 1/\tau$ gives the desired $w \in A_p$.

4. With the above results and the theorems in **[2, 6]** it is possible to characterize those u which are bounded above by a $w \in A_p$, and those v which are bounded below by a $w \in A_p$.

THEOREM 4. (1) *Given* $u \geqslant 0$. *Then there is a* $w \in A_p$ *with* $cu \leqslant w$ *if and only if there is* $\tau > 1$ *such that*

$$\int_{\mathbf{R}^n} \frac{u^\tau(x)}{(1 + |x|^n)^p}\, dx < \infty.$$

(2) *Given* $v > 0$. *Then there is* $w \neq 0$ *in* A_p *such that* $w \leqslant cv$ *if and only if there is* $\tau > 1$ *such that* $\sup_{r>0} r^{-np'}\int_{|x| \leqslant r} v^{\tau(1-p')}\, dx < \infty$.

PROOF. We prove (1) and assume first that $cu \leqslant w$, $w \in A_p$. Then there is $\tau > 1$ with $w^\tau \in A_p$ and thus

$$\int_{\mathbf{R}^n} \frac{u^\tau(x)}{(1 + |x|^n)^p} dx < \infty .$$

Conversely, if the integral is finite, we have v_0 so that $(u^\tau, v_0) \in A_p$. If we let $v = v_0^{1/\tau}$, then $(u^\tau, v^\tau) \in A_p$, and Theorem 3 completes the proof.

The proof of (2) is exactly the same as above using instead [**2**].

5. It is natural to inquire what one can say about a pair of functions (u, v) for which there is a function w so that:

(i) $c_1 u \leqslant w \leqslant c_2 v$.

(ii) $(u, w) \in A_p$, $(w, v) \in A_p$.

We use the methods of the papers [**3, 4**] to investigate this question.

Let $f_\sigma^*(t)$ be the nonincreasing rearrangement of f: $\mathbf{R}^n \to \mathbf{R}$ with respect to the measure σ on $\mathbf{R}^n$. If for a pair (u, v), $d\mu = u dx$, $d\nu = v dx$, and

$$\Phi(t) \equiv \Phi_{u,v}(t) = \sup_Q \frac{\mu(Q)}{|Q|} \left(\frac{\chi_Q}{v} \right)_\nu^* (\mu(Q)t),$$

then by Theorem 1 of [**3**],

$$(1) \qquad (Mf)_\mu^*(\xi) \leqslant A \int_0^\infty \Phi(t) f_\nu^*(t\xi)\, dt.$$

If $(u, v) \in A_p$, we have shown that $\Phi \in L(p', s)$ for some $1 \leqslant s \leqslant \infty$, and any such s can occur [**3**, Theorems 3, 4].

Suppose now we have (i) and (ii) above. Let $\Phi_1 = \Phi_{u,w}$, $\Phi_2 = \Phi_{w,v}$, and let $M_2 f = M\{Mf\}$. A twofold application of (1) yields with $d\lambda = w dx$,

$$\begin{aligned}(M_2 f)_\mu^*(\xi) &\leqslant A \int_0^\infty \Phi_1(t)(Mf)_\lambda^*(t\xi)\, dt \\ &\leqslant A^2 \int_0^\infty \int_0^\infty \Phi_1(t)\Phi_2(\tau) f_\nu^*(\tau t \xi)\, d\tau\, dt \\ &= A^2 \int_0^\infty \int_0^\infty \frac{1}{t} \Phi_1(t) \Phi_2\left(\frac{u}{t}\right) f_\nu^*(u\xi)\, dt\, du.\end{aligned}$$

Hence

$$(2) \qquad (M_2 f)_\mu^* \leqslant c \int_0^\infty \Psi(u) f_\nu^*(u\xi)\, du,$$

where $\Psi(u) = \int_0^\infty \Phi_1(t)\Phi_2(u/t)\, dt/t$.

The key to our question lies in the study of $\Psi(u)$ in terms Φ_1, Φ_2. For f, g: $\mathbf{R}^+ \to \mathbf{R}^+$, define $h(u) = \int_0^\infty (1/t) f(t) g(u/t)\, dt$, and write for ϕ: $\mathbf{R}^+ \to \mathbf{R}^+$,

$$\|\phi\|_{p,s} = \left\{ \int_0^\infty \left[t^{1/p} \phi(t) \right]^s \frac{dt}{t} \right\}^{1/s}.$$

If ϕ is the nonincreasing rearrangement of a function f, this is the familiar $L(p, s)$-norm of f.

LEMMA 3. *Let* $1 \leqslant p < \infty$, $1/s + 1/\sigma \geqslant 1$, $1/r = 1/s + 1/\sigma - 1$. *Then* $\|h\|_{p,r} \leqslant \|f\|_{p,s}\|g\|_{p,\sigma}$.

PROOF. We note that $1/r + 1/\sigma' + 1/s' = 1$, and we write

$$f(t)g\left(\frac{u}{t}\right) = \left[t^{(s-\sigma)/pr}f(t)^{s/r}g\left(\frac{u}{t}\right)^{\sigma/r}\right]\cdot\left[t^{s/p\sigma'}f(t)^{s/\sigma'}\right]\cdot\left[t^{-\sigma/ps'}g\left(\frac{u}{t}\right)^{\sigma/s'}\right].$$

By Hölder's inequality with respect to the measure dt/t we get

$$h(u) \leqslant \left(\int_0^\infty t^{(s-\sigma)/p}f(t)^s g\left(\frac{u}{t}\right)^\sigma \frac{dt}{t}\right)^{1/r}\cdot\left(\int_0^\infty t^{s/p}f(t)^s\frac{dt}{t}\right)^{1/\sigma'}$$
$$\cdot\left(\int_0^\infty t^{-\sigma/p}g\left(\frac{u}{t}\right)^\sigma\frac{dt}{t}\right)^{1/s'}.$$

The last integral is $u^{-\sigma/ps'}\|g\|_{p,\sigma}^{\sigma/s'}$, and the middle term is $\|f\|_{p,s}^{s/\sigma'}$. Now raise both sides to the rth power, multiply by $u^{r/p-1}$ and integrate with respect to u. We obtain

$$\|h\|_{p,r}^r \leqslant \left(\int_0^\infty t^{(s-\sigma)/p}f(t)^s\int_0^\infty u^{r/p-1-\sigma r/ps'}g\left(\frac{u}{t}\right)^\sigma du\frac{dt}{t}\right)\cdot\|g\|_{p,\sigma}^{\sigma r/s'}\cdot\|f\|_{p,s}^{sr/\sigma'}.$$

The integral in u, since $1 - \sigma/s' = \sigma/r$, becomes

$$\int_0^\infty u^{\sigma/p}g\left(\frac{u}{t}\right)^\sigma\frac{du}{u} = t^{\sigma/p}\cdot\|g\|_{p\sigma}^\sigma,$$

and hence the first factor is

$$\int_0^\infty t^{(s-\sigma)/p}\cdot t^{\sigma/p}f(t)^s\frac{dt}{t} = \|f\|_{p,s}^s.$$

If we extract the rth root we obtain the desired norm inequality.

REMARK. This is a "Young's type" convolution inequality applied to the second index in the $L(p,q)$-spaces and our proof is a suitable adaptation of the usual proof for convolutions [7, p. 37].

The next lemma is needed for the integrals of the type (1), (2) for the maximal function.

LEMMA 4. *For* $f, g\colon \mathbf{R}^+ \to \mathbf{R}^+$, *let* $F(u) = \int_0^\infty f(t)g(tu)\,dt$. *If* $1 < p < \infty$, $1/s + 1/\sigma \geqslant 1$, $1/r = 1/s + 1/\sigma - 1$, *then*

$$\|F\|_{p,r} \leqslant \|f\|_{p',s}\|g\|_{p,\sigma}.$$

PROOF. If we let $t = 1/\tau$, then

$$F(u) = \int_0^\infty \frac{f(1/\tau)}{\tau}g\left(\frac{u}{\tau}\right)\frac{d\tau}{\tau} \equiv \int_0^\infty f_1(\tau)g\left(\frac{u}{\tau}\right)\frac{d\tau}{\tau}.$$

Thus by Lemma 1, $\|F\|_{p,r} \leqslant \|f_1\|_{p,s}\|g\|_{p,\sigma}$, and

$$\|f_1\|_{p,s}^s = \int_0^\infty\left[\tau^{1/p-1}f\left(\frac{1}{\tau}\right)\right]^s\frac{d\tau}{\tau} = \int_0^\infty \tau^{-s/p'-1}f\left(\frac{1}{\tau}\right)^s d\tau$$
$$= \int_0^\infty t^{s/p'}f(t)^s\frac{dt}{t} = \|f\|_{p',s}^s.$$

6. We apply now Lemmas 3 and 4 to our original problem of finding a w between u, v so that $(u, w) \in A_p$, $(w, v) \in A_p$.

First we state Lemma 4 in the context of $(Mf)^*_\mu(\xi) \leqslant A\int_0^\infty \Phi(t) f^*_\nu(t\xi)\,dt$.

THEOREM 5. *If* $1/s + 1/\sigma \geqslant 1$, $1/r = 1/s + 1/\sigma - 1$, *then* $\|Mf\|_{p,r,\mu} \leqslant c\|\Phi\|_{p',s}\|f\|_{p,\sigma,\nu}$.

THEOREM 6. *Let* (u, v) *be a pair of functions for which there is a function* w *such that* $\Phi_1 = \Phi_{u,w} \in L(p', s)$, $\Phi_2 = \Phi_{w,v} \in L(p', \sigma)$, $1/s + 1/\sigma + 1/\rho \geqslant 2$, *and* $1/\tau = 1/s + 1/\sigma + 1/\rho - 2$. *Then*

$$\|M_2 f\|_{p,\tau,\mu} \leqslant c\|\Phi_1\|_{p',s} \cdot \|\Phi_2\|_{p',\sigma} \cdot \|f\|_{p,\rho,\nu}.$$

PROOF. We have seen that

$$(M_2 f)^*_\mu(\xi) \leqslant c\int_0^\infty \Psi(u) f^*_\nu(u\xi)\,du,$$

where $\Psi(u) = \int_0^\infty \Phi_1(t)\Phi_2(u/t)\,dt/t$. If $1/r = 1/s + 1/\sigma - 1$, then by Lemma 3,

$$\|\Psi\|_{p',r} \leqslant \|\Phi_1\|_{p',s}\|\Phi_2\|_{p',\sigma}.$$

Finally, since $1/\tau = 1/r + 1/\rho - 1$, Lemma 4 gives $\|M_2 f\|_{p,\tau,\mu} \leqslant c\|\Psi\|_{p',r}\|f\|_{p,\rho,\nu}$, and the proof is complete.

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