INSERTING $A_p$-WEIGHTS

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Abstract. Necessary and sufficient conditions are given for inserting a single weight $w \in A_p$ between $u$ and $v$, i.e., $c_1 u \leq w \leq c_2 v$.

1. Let $(u, v)$ be a pair of nonnegative measurable functions on $\mathbb{R}^n$. We ask under what conditions it is possible to find a function $w$ with $c_1 u \leq w \leq c_2 v$ so that both pairs $(u, w)$, $(w, v)$ are in $A_p$, or even better $w$ itself is in $A_p$. We recall that $w \in A_p$ iff $(\int_Q w^q)_{p-1}^{1/p} \leq C \|Q\|_p$, and $(u, v) \in A_p$ iff $(\int_Q u^q)_{p-1}^{1/p} \leq C \|Q\|_p$. These classes were introduced by Muckenhoupt [5] and play an important role in weighted norm inequalities for many operators, as the Hardy-Littlewood maximal operator $Mf(x) = \sup_{r>0} \frac{1}{r^n} \int_{B(x,r)} |f(y)| dy$, the Hilbert transform, and many others.

The single weight problem is easier to handle than the double weight problem, and thus the weighted norm inequalities for $(u, v)$ would follow from those of $w \in A_p$ if $c_1 u \leq w \leq c_2 v$.

2. Throughout $1 < p < \infty$, unless otherwise noted. We need the following lemmas.

Lemma 1. $(u, v) \in A_p$ implies $(v^{1-p}, u^{1-p}) \in A_{p'}$, $1/p + 1/p' = 1$.

Proof. This is simply $(1 - p')(1 - p) = 1$.

Lemma 2. Let $(u, v) \in A_p$ and let $0 < \delta < 1$. If $(q - 1)/(p - 1) = \delta$, then $(u^\delta, v^\delta) \in A_q$.

Proof. Note that $1 < q < p$. Since $(v^{1-p'}, u^{1-p'}) \in A_{p'}$ and $q' > p'$, we have $(v^{1-p}, u^{1-p}) \in A_q$. We use Lemma 1 again and infer that $(u^{1-p'(1-q)}, v^{1-p'(1-q)}) \in A_q$, that is $(u^\delta, v^\delta) \in A_q$.

Corollary. If $(u, v) \in A_p$, $0 < \delta < 1$, then $\|Mf\|_{p, u^\delta} \leq A_\delta \|f\|_{p, v^\delta}$.

Proof. Since $(u^\delta, v^\delta) \in A_q$ and $p > \delta(p - 1) + 1 = q$, we get $u^\delta(Mf > y) \leq c \|f\|_{q, u^\delta}^{1/q} y^q$ [5], and thus $\|Mf\|_{p, u^\delta} \leq A_\delta \|f\|_{p, v^\delta}$ by the Marcinkiewicz interpolation theorem.

3. The following theorem is the key to our problem. Its proof is a two weight adaptation of the Coifman, Jones and Rubio de Francia argument of factoring an $A_p$-weight [1].

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Theorem 1. Assume that

(i) \( \|Mf\|_{p,u} \leq A\|f\|_{p,v} \),

(ii) \( \|Mf\|_{p',e^{1-p}} \leq B\|f\|_{p',e^{1-p}} \).

Then there are functions \( w_j \geq 0 \) such that \( u^{1/p}Mw_j \leq C_j w_j v^{1/p}, j = 1,2, \) and \( u^{1/p}v^{1/p} = w_j w_2^{1-p} \).

Proof. We may assume that \( p \geq 2 \) (if \( 1 < p < 2 \) simply interchange below the role of (i) and (ii)). Let \( s = p/p' \), and \( M_s f = M(f^s)^{1/s} \). The operator \( S(f) = u^{1/p}M(fv^{-1/p}) + v^{-1/p}M_s(fu^{1/p}) \) satisfies \( \|S(f)\|_p \leq C\|f\|_p \). We let \( \overline{C} > C \), and for \( u_0 \in L^p, u_0 > 0 \), let \( U = \Sigma \overline{S}(u_0)/\overline{C}^j \). Then \( U \in L^p \) and \( S(U) \leq \overline{C} U \). Hence

\[
M(Uv^{-1/p})u^{1/p} \leq S(U) \leq \overline{C} U \quad \text{and} \quad v^{-1/p}M_s(Uu^{1/p}) \leq \overline{C} U.
\]

Thus, if \( w_2 = Uv^{-1/p} \), we get \( u^{1/p}Mw_2 \leq c_2 w_2 v^{1/p} \), and if we let \( w_1 = U^{1/p} \), we get \( u^{1/p}Mw_1 \leq c_1 w_1 v^{1/p} \), and \( u^{1/p}v^{1/p} = w_1 w_2^{1-p} \).

Theorem 2. Let \((u,v) \in A_p \) and \( 0 < \delta < 1 \). Then there exists \( w = w_\delta \in A_p \) such that \( c_1 u^\delta \leq w \leq c_2 v^\delta \).

Proof. Choose \( 0 < \varepsilon, \eta < 1 \) such that \( \delta = \varepsilon \eta \). From the corollary we know that \( \|Mf\|_{p,v} \leq A\|f\|_{p,v}, \|Mf\|_{p',e^{1-p}} \leq B\|f\|_{p',e^{1-p}} \). Thus by Theorem 1, \( u^{1/p}v^{1/p} = w_1 w_2^{1-p} \), where \( Mw_j \leq c_j w_j (v/u)^{1/p} \). Note that

\[
v^\varepsilon = w_1 \left( \frac{v}{u} \right)^{1/p} w_2^{1-p} \geq cMw_1 Mw_2^{1-p} \geq cw_1 \cdot w_2^{1-p} \left( \frac{v}{u} \right)^{(1-p)/p} = cu^\varepsilon,
\]

and thus \( c_1 u^\delta \leq (Mw_1)^\varepsilon(Mw_2)^{(1-p)\varepsilon} \leq c_2 v^\delta \). Since \( 0 < \eta < 1 \), \((Mw_j)^\eta \in A_1 \), a result due to Coifman. It is easy to see that \( w = (Mw_1)^\eta(Mw_2)^{(1-p)\eta} \in A_p \).

We are now able to give necessary and sufficient conditions for inserting \( A_p \)-weights.

Theorem 3. Let \((u,v) \) be a pair of nonnegative functions. Then there exists \( w \in A_p \) with \( c_1 u \leq w \leq c_2 v \) if and only if there is \( \tau > 1 \) such that \((u^\tau, v^\tau) \in A_p \).

Proof. Since \( w \) and \( w^{1-p} \) obey a reverse Hölder inequality [5], there is \( \tau > 1 \) such that \( w^\tau \in A_p \), and thus \((u^\tau, v^\tau) \in A_p \). Conversely, if \((u^\tau, v^\tau) \in A_p \), then Theorem 2 with \( \delta = 1/\tau \) gives the desired \( w \in A_p \).

4. With the above results and the theorems in [2,6] it is possible to characterize those \( u \) which are bounded above by a \( w \in A_p \), and those \( v \) which are bounded below by a \( w \in A_p \).

Theorem 4. (1) Given \( u \geq 0 \). Then there is a \( w \in A_p \) with \( cu \leq w \) if and only if there is \( \tau > 1 \) such that

\[
\int_{\mathbb{R}^n} \frac{u^\tau(x)}{(1 + |x|^{\tau})} dx < \infty.
\]

(2) Given \( v > 0 \). Then there is \( w \neq 0 \) in \( A_p \) such that \( w \leq cv \) if and only if there is \( \tau > 1 \) such that \( \sup_{r>0} r^{-\tau} \int_{|x|<r} v^\tau(1-p) dx < \infty \).
Proof. We prove \( (1) \) and assume first that \( cu \leq w, w \in A_p \). Then there is \( \tau > 1 \) with \( w^\tau \in A_p \) and thus

\[
\int \frac{u^\tau(x)}{(1 + |x|^\eta)^p} \, dx < \infty.
\]

Conversely, if the integral is finite, we have \( v_0 \) so that \( (u^\tau, v_0) \in A_p \). If we let \( v = v_0^{1/\tau} \), then \( (u^\tau, v^\tau) \in A_p \) and Theorem 3 completes the proof.

The proof of \( (2) \) is exactly the same as above using instead \([2]\).

5. It is natural to inquire what one can say about a pair of functions \( (u, v) \) for which there is a function \( w \) so that:

\( (i) \) \( cu \leq w \leq c_2 v \).

\( (ii) (u, w) \in A_p, (w, v) \in A_p \).

We use the methods of the papers \([3, 4]\) to investigate this question.

Let \( f^*(\tau) \) be the nonincreasing rearrangement of \( f : \mathbb{R}^n \to \mathbb{R} \) with respect to the measure \( \sigma \) on \( \mathbb{R}^n \). If for a pair \( (u, v) \), \( d\mu = u \, dx, d\nu = v \, dx \), and

\[
\Phi(t) \equiv \Phi_{u,v}(t) = \sup_{Q} \frac{\mu(Q)}{|Q|} \left( \frac{X_{\nu} \psi}{v} \right)_r (\mu(Q)t),
\]

then by Theorem 1 of \([3]\),

\[
(Mf)^*_p(\xi) \leq A \int_0^\infty \Phi(t) f^*_p(t\xi) \, dt.
\]

If \( (u, v) \in A_p \), we have shown that \( \Phi \in L(p', s) \) for some \( 1 \leq s \leq \infty \), and any such \( s \) can occur \([3, \text{Theorems } 3, 4]\).

Suppose now we have \( (i) \) and \( (ii) \) above. Let \( \Phi_1 = \Phi_{u,v} \), \( \Phi_2 = \Phi_{v,u} \), and let \( M_2 f = M\{Mf\} \). A twofold application of \( (1) \) yields with \( d\lambda = w \, dx \).

\[
(M_2 f)_p^*(\xi) \leq A \int_0^\infty \Phi_1(t)(Mf)_p^*(t\xi) \, dt
\]

\[
\leq A^2 \int_0^\infty \int_0^\infty \Phi_1(t)\Phi_2(\tau)f^*_p(\tau t\xi) \, d\tau dt
\]

\[
= A^2 \int_0^\infty \int_0^\infty \frac{1}{\tau} \Phi_1(t)\Phi_2(\tau)\left( \frac{u}{\tau} \right)_r f^*_p(u\xi) \, d\tau du.
\]

Hence

\[
(M_2 f)_p^* \leq c \int_0^\infty \Psi(u)f^*_p(u\xi) \, du.
\]

where \( \Psi(u) = \int_0^\infty \Phi_1(t)\Phi_2(u/t) \, dt/t \).

The key to our question lies in the study of \( \Psi(u) \) in terms \( \Phi_1, \Phi_2 \). For \( f, g: \mathbb{R}^+ \to \mathbb{R}^+ \), define \( h(u) = \int_0^\infty (1/t)f(t)g(u/t) \, dt \), and write for \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \).

\[
\|\phi\|_{p,s} = \left( \int_0^\infty \left[ t^{1/p} \phi(t) \right]^s \, dt \right)^{1/s}.
\]

If \( \phi \) is the nonincreasing rearrangement of a function \( f \), this is the familiar \( L(p, s) \)-norm of \( f \).
Lemma 3. Let \( 1 \leq p < \infty, 1/s + 1/\sigma \geq 1, 1/r = 1/s + 1/\sigma - 1 \). Then \( \| h \|_{p,r} \leq \| f \|_{p,s} \| g \|_{p,\sigma} \).

Proof. We note that \( 1/r + 1/\sigma' + 1/s' = 1 \), and we write
\[
(I(t)g(t))g(t) = (t^{s'} \cdot \sigma')^{1/s'} \cdot (t^{\sigma'})^{1/\sigma'} \cdot (t^{s'})^{1/s'}.
\]

By Hölder's inequality with respect to the measure \( dt/t \) we get
\[
\frac{1}{r} \left( \int_0^\infty (t^{s'} \cdot \sigma')^{1/s'} \cdot (t^{\sigma'})^{1/\sigma'} \cdot (t^{s'})^{1/s'} \right)^{1/r} \cdot \left( \int_0^\infty t^{1/r} f(t) \right)^{1/\sigma'} \cdot \left( \int_0^\infty t^{s/r} g(t) \right)^{1/s'}.
\]

The last integral is \( u^{-\sigma'/p} \| g \|_{p,\sigma'}^{1/s'} \), and the middle term is \( \| f \|_{p,s'}^{1/s'} \). Now raise both sides to the \( r \)th power, multiply by \( u^{r/p - 1} \) and integrate with respect to \( u \). We obtain
\[
\| h \|_{p,r}^r \leq \left( \int_0^\infty t^{r^{s'}} \cdot \sigma')^{1/s'} \cdot (t^{\sigma'})^{1/\sigma'} \cdot (t^{s'})^{1/s'} \right)^{1/r} \cdot \left( \int_0^\infty t^{s/r} g(t) \right)^{1/s'} \cdot \| g \|_{p,\sigma'}^{1/s'} \cdot \| f \|_{p,s'}^{1/s'}.
\]

The integral in \( u \), since \( 1 - \sigma/s' = \sigma/r \), becomes
\[
\int_0^\infty u^{\sigma/p} g(t) \frac{u}{t} \frac{dt}{u} = t^{\sigma/p} \cdot \| g \|_{p,\sigma},
\]
and hence the first factor is
\[
\int_0^\infty t^{(r-1)/p} \cdot t^{\sigma/p} f(t) \frac{dt}{u} = \| f \|_{p,s}.
\]

If we extract the \( r \)th root we obtain the desired norm inequality.

Remark. This is a "Young's type" convolution inequality applied to the second index in the \( L(p, q) \)-spaces and our proof is a suitable adaptation of the usual proof for convolutions [7, p. 37].

The next lemma is needed for the integrals of the type (1), (2) for the maximal function.

Lemma 4. For \( f, g : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \), let \( F(u) = \int_0^\infty f(t)g(tu) \frac{dt}{u} \). If \( 1 < p < \infty, 1/s + 1/\sigma \geq 1, 1/r = 1/s + 1/\sigma - 1 \), then
\[
\| F \|_{p,r} \leq \| f \|_{p,s} \| g \|_{p,\sigma}.
\]

Proof. If we let \( t = 1/\tau \), then
\[
F(u) = \int_0^\infty \frac{f(1/\tau)}{\tau} g \left( \frac{u}{\tau} \right) \frac{d\tau}{\tau} = \int_0^\infty f_1(\tau) g \left( \frac{u}{\tau} \right) \frac{d\tau}{\tau}.
\]

Thus by Lemma 1, \( \| F \|_{p,r} \leq \| f_1 \|_{p,s} \| g \|_{p,\sigma} \), and
\[
\| f_1 \|_{p,s}^s = \int_0^\infty \frac{1}{\tau^{1/p-1}} f \left( \frac{1}{\tau} \right) \frac{d\tau}{\tau} = \int_0^\infty \frac{1}{\tau^{1/s-1}} f \left( \frac{1}{\tau} \right) \frac{d\tau}{\tau} = \int_0^\infty t^{s-1} f(t) \frac{dt}{t} = \| f \|_{p,s}^s.
\]
6. We apply now Lemmas 3 and 4 to our original problem of finding a \( w \) between \( u, v \) so that \( (u, w) \in A_p, (w, v) \in A_p \).

First we state Lemma 4 in the context of \( (Mf)_{n+1}^n(\xi) \leq A f_0^\infty \Phi(t) f^n_\infty(t\xi) \ dt \).

**Theorem 5.** If \( 1/s + 1/\sigma \geq 1, 1/r = 1/s + 1/\sigma - 1 \), then \( \| Mf \|_{p', r, \mu} \leq c\| \Phi \|_{p', s} \| f \|_{p, \sigma, \mu} \).

**Theorem 6.** Let \((u, v)\) be a pair of functions for which there is a function \( w \) such that 
\( \Phi_1 = \Phi_{u,w} \in L(p', s), \Phi_2 = \Phi_{w,v} \in L(p', \sigma), 1/s + 1/\sigma + 1/\rho \geq 2, \) and \( 1/\tau = 1/s + 1/\sigma + 1/\rho - 2 \). Then
\[
\| M_2 f \|_{p', \tau, \mu} \leq c\| \Phi_1 \|_{p', s} \| \Phi_2 \|_{p', \sigma} \| f \|_{p, \sigma, \mu}.
\]

**Proof.** We have seen that
\[
(M_2 f)_{n+1}^n(\xi) \leq c \int_0^\infty \Psi(u) f_\infty^n(u\xi) \ du,
\]
where \( \Psi(u) = \int_0^\infty \Phi_1(t) \Phi_2(u/t) \ dt/t \). If \( 1/r = 1/s + 1/\sigma - 1 \), then by Lemma 3,
\[
\| \Psi \|_{p', r} \leq \| \Phi_1 \|_{p', s} \| \Phi_2 \|_{p', \sigma}.
\]
Finally, since \( 1/\tau = 1/r + 1/\rho - 1 \), Lemma 4 gives \( \| M_2 f \|_{p', \tau, \mu} \leq c\| \Psi \|_{p', r} \| f \|_{p, \rho, \sigma} \), and the proof is complete.

**References**


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