INSERTING $A_p$-WEIGHTS

C. J. NEUGEBAUER

Abstract. Necessary and sufficient conditions are given for inserting a single weight $w \in A_p$ between $u$ and $v$, i.e., $c_1u \leq w \leq c_2v$.

1. Let $(u, v)$ be a pair of nonnegative measurable functions on $\mathbb{R}^n$. We ask under what conditions is it possible to find a function $w$ with $c_1u \leq w \leq c_2v$ so that both pairs $(u, w)$, $(w, v)$ are in $A_p$, or even better $w$ itself is in $A_p$. We recall that $w \in A_p$ iff $(\int Q|w(\cdot)|^{p'-1}Q)^{1/p'} \leq C |Q|_p$, and $(u, v) \in A_p$ iff $(\int Q|u(\cdot)|^{p-1}Q)^{1/p} \leq C |Q|_p$. These classes were introduced by Muckenhoupt [5] and play an important role in weighted norm inequalities for many operators, as the Hardy-Littlewood maximal operator $Mf(x) = \sup_{Q} |Q|^{-1/q} \int f(x) \, dt$, the Hilbert transform, and many others.

The single weight problem is easier to handle than the double weight problem, and thus the weighted norm inequalities for $(u, v)$ would follow from those of $w \in A_p$ if $c_1u \leq w \leq c_2v$.

2. Throughout $1 < p < \infty$, unless otherwise noted. We need the following lemmas.

**Lemma 1.** Let $(u, v) \in A_p$ implies $(v^{-p'}, u^{-p'}) \in A_{p'}, 1/p + 1/p' = 1$.

**Proof.** This is simply $(1 - p')(1 - p) = 1$.

**Lemma 2.** Let $(u, v) \in A_p$ and let $0 < \delta < 1$. If $(q - 1)/(p - 1) = \delta$, then $(u^\delta, v^\delta) \in A_q$.

**Proof.** Note that $1 < q < p$. Since $(v^{-p'}, u^{-p'}) \in A_{p'}$ and $q' > p'$, we have $(v^{-p'}, u^{-p'}) \in A_{q'}$. We use Lemma 1 again and infer that $(u^{(1-p')(1-q')}, v^{(1-p')(1-q')}) \in A_{q'}$, that is $(u^\delta, v^\delta) \in A_q$.

**Corollary.** If $(u, v) \in A_p$, $0 < \delta < 1$, then $\|Mf\|_{p, u^\delta} \leq A_{\delta} \|f\|_{p, v^\delta}$.

**Proof.** Since $(u^\delta, v^\delta) \in A_q$ and $p > 0$, we get $u^\delta(Mf > y) \leq c \|f\|^q_{p, v^\delta} / y^q$ [5], and thus $\|Mf\|_{p, u^\delta} \leq A_{\delta} \|f\|_{p, v^\delta}$ by the Marcinkiewicz interpolation theorem.

3. The following theorem is the key to our problem. Its proof is a two weight adaptation of the Coifman, Jones and Rubio de Francia argument of factoring an $A_p$-weight [1].

Received by the editors June 15, 1982.

1980 Mathematics Subject Classification. Primary 42B25.
Theorem 1. Assume that
(i) \[ \| Mf \|_{p,w} \leq A \| f \|_{p,v}, \]
(ii) \[ \| Mf \|_{p,\varepsilon^{-p}} \leq B \| f \|_{p,\varepsilon^{-p}}. \]
Then there are functions \( w_j \geq 0 \) such that \( u^{1/p}Mw_j \leq C_jw_jv^{1/p}, j = 1,2, \) and \( u^{1/p}v^{1/p} = w_1w_2^{1-p}. \)

Proof. We may assume that \( p \geq 2 \) (if \( 1 < p < 2 \) simply interchange below the role of (i) and (ii)). Let \( s = p/p' \), and \( M_s f = M(f^{1/s}) \). The operator \( S(f) = u^{1/p}M(fv^{-1/p}) + v^{-1/p}M_s(fu^{1/p}) \) satisfies \( \| S(f) \|_p \leq C \| f \|_p. \) We let \( \tilde{C} > C, \) and for \( u_0 \in L^p, u_0 > 0, \) let \( U = \sum_{j=1}^\infty S(u_0)/C_j. \) Then \( U \in L^p \) and \( S(U) \leq \tilde{C}U. \) Hence
\[
M(Uv^{-1/p})u^{1/p} \leq S(U) \leq \tilde{C}U \quad \text{and} \quad v^{-1/p}M_s(Uu^{1/p}) \leq \tilde{C}U.
\]
Thus, if \( w_2 = Uv^{-1/p}, \) we get \( u^{1/p}Mw_2 \leq c_2w_2v^{1/p}, \) and if we let \( w_1 = U^{-1/p} \), we get \( u^{1/p}Mw_1 \leq c_1w_1v^{1/p}, \) and \( u^{1/p}v^{1/p} = w_1w_2^{1-p}. \)

Theorem 2. Let \((u,v) \in A_p \) and \( 0 < \delta < 1. \) Then there exists \( w = w_\delta \in A_p \) such that \( c_1u^\delta \leq w \leq c_2v^\delta. \)

Proof. Choose \( 0 < \varepsilon, \eta < 1 \) such that \( \delta = \varepsilon\eta. \) From the corollary we know that \( \| Mf \|_{p,v} \leq A \| f \|_{p,v}, \) \( \| Mf \|_{p,\varepsilon^{-p}} \leq B \| f \|_{p,\varepsilon^{-p}}. \) Thus by Theorem 1, \( u^{r/p}v^{r/p} = w_1w_2^{1-p}, \) where \( Mw_j \leq c_jw_j(u/v)^{1/p}. \) Note that
\[
v^\varepsilon = w_1\left(\frac{v}{u}\right)^{r/p}w_2^{1-p} \geq cMw_1(Mw_2)^{1-p} \geq cw_1w_2^{1-p} = cu^\varepsilon,
\]
and thus \( c_1u^\delta \leq (Mw_1)^\eta(Mw_2)^{\eta(1-p)} \leq c_2v^\delta. \) Since \( 0 < \eta < 1, (Mw_j)^\eta \in A_1, \) a result due to Coifman. It is easy to see that \( w = (Mw_1)^\eta(Mw_2)^{\eta(1-p)} \in A_p. \)

We are now able to give necessary and sufficient conditions for inserting \( A_p \)-weights.

Theorem 3. Let \((u,v)\) be a pair of nonnegative functions. Then there exists \( w \in A_p \) with \( c_1u \leq w \leq c_2v \) if and only if there is \( \tau > 1 \) such that \((u^\tau,v^\tau) \in A_p.\)

Proof. Since \( w \) and \( w^{1-p} \) obey a reverse Hölder inequality [5], there is \( \tau > 1 \) such that \( w^\tau \in A_p, \) and thus \((u^\tau,v^\tau) \in A_p. \) Conversely, if \((u^\tau,v^\tau) \in A_p, \) then Theorem 2 with \( \delta = 1/\tau \) gives the desired \( w \in A_p. \)

4. With the above results and the theorems in [2,6] it is possible to characterize those \( u \) which are bounded above by a \( w \in A_p, \) and those \( v \) which are bounded below by a \( w \in A_p. \)

Theorem 4. (1) Given \( u \geq 0. \) Then there is a \( w \in A_p \) with \( cu \leq w \) if and only if there is \( \tau > 1 \) such that
\[
\int_{\mathbb{R}^n} \frac{u^\tau(x)}{(1 + |x|^\tau)^n} dx < \infty.
\]
(2) Given \( v > 0. \) Then there is \( w \neq 0 \) in \( A_p \) such that \( w \leq cv \) if and only if there is \( \tau > 1 \) such that \( \sup_{r > 0} r^{-n} \int_{|x| < r} u^{\tau(1-p)} v^{\tau(1-p)} dx < \infty. \)
PROOF. We prove (1) and assume first that \( cu \leq w \leq c_2 v \). Then there is \( \tau > 1 \) with \( w^\tau \in \mathbb{A}_p \) and thus

\[
\int_{\mathbb{R}^n} \frac{u^\tau(x)}{(1 + |x|^\sigma)^\rho} \, dx < \infty.
\]

Conversely, if the integral is finite, we have \( v_0 \) so that \( (u^\tau, v_0) \in \mathbb{A}_p \). If we let \( v = v_0^1/\tau \), then \( (u^\tau, v^\tau) \in \mathbb{A}_p \) and Theorem 3 completes the proof.

The proof of (2) is exactly the same as above using instead [2].

5. It is natural to inquire what one can say about a pair of functions \((u, v)\) for which there is a function \( w \) so that:

\( (i) \ c_1 u \leq w \leq c_2 v \).

\( (ii) \ (u, w) \in \mathbb{A}_p, (w, v) \in \mathbb{A}_p \).

We use the methods of the papers \([3, 4]\) to investigate this question.

Let \( f^*(t) \) be the nonincreasing rearrangement of \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) with respect to the measure \( \sigma \) on \( \mathbb{R}^n \). If for a pair \((u, v)\), \( d\mu = u \, dx, d\nu = v \, dx \), and

\[
\Phi(t) = \Phi_{u, c}(t) = \sup_{Q} \frac{\mu(Q)}{|Q|} \left( \frac{\chi_{Q}}{v} \right)^\tau, \quad (\mu(Q)c(t)),
\]

then by Theorem 1 of \([3]\),

\[
(1)
\]

\[
(M f)^*(\xi) \leq A \int_{0}^{\infty} \Phi(t) f^*(t \xi) \, dt.
\]

If \((u, v) \in \mathbb{A}_p \), we have shown that \( \Phi \in L(\rho, s) \) for some \( 1 \leq s < \infty \), and any such \( s \) can occur \([3, \text{Theorems 3, 4}]\).

Suppose now we have (i) and (ii) above. Let \( \Phi_1 = \Phi_{u, c}, \Phi_2 = \Phi_{u, c^2} \), and let \( M_2 f = M\{M f\} \). A twofold application of (1) yields with \( d\lambda = w \, dx \).

\[
(M_2 f)^*(\xi) \leq A \int_{0}^{\infty} \Phi_1(t) (M f)^*(t \xi) \, dt
\]

\[
\leq A^2 \int_{0}^{\infty} \Phi_1(t) \Phi_2(\tau) f^*(\tau \xi) \, d\tau \, dt
\]

\[
= A^2 \int_{0}^{\infty} \int_{0}^{\infty} \frac{1}{r} \Phi_1(t) \Phi_2 \left( \frac{u}{r} \right) f^*(u \xi) \, dt \, du.
\]

Hence

\[
(2)
\]

\[
(M_2 f)^*(\xi) \leq c \int_{0}^{\infty} \Psi(v) f^*(u \xi) \, du.
\]

where \( \Psi(u) = \int_{0}^{\infty} \Phi_1(t) \Phi_2(u/t) \, dt/t \).

The key to our question lies in the study of \( \Psi(u) \) in terms \( \Phi_1, \Phi_2 \). For \( f, g: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \), define \( h(u) = \int_{0}^{u} (1/t) f(t) g(u/t) \, dt \), and write for \( \phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \).

\[
\| \phi \|_{p, s} = \left\{ \int_{0}^{\infty} \left[ t^{1/p} \phi(t) \right]^s \, dt \right\}^{1/s}.
\]

If \( \phi \) is the nonincreasing rearrangement of a function \( f \), this is the familiar \( L(p, s) \)-norm of \( f \).
Lemma 3. Let $1 < p < \infty$, $1/s + 1/\alpha \geq 1$, $1/r = 1/s + 1/\alpha - 1$. Then $\|h\|_{p,r} \leq \|f\|_{p,s} \|g\|_{p,\alpha}$.

Proof. We note that $1/r + 1/\alpha' + 1/s' = 1$, and we write

$$f(t)g\left(\frac{u}{t}\right) = \left[t^{(s-\alpha)/p}f(t)\right]^{1/r} \|g\|_{p,\alpha'} \left[t^{\alpha/p}g\left(\frac{u}{t}\right)\right]^{-1/\alpha'}.\left[t^{s/\alpha'}g\left(\frac{u}{t}\right)\right]^{1/\alpha'}.\left[t^{s/\alpha'}g\left(\frac{u}{t}\right)\right].$$

By Hölder’s inequality with respect to the measure $dt/t$ we get

$$h(u) \leq \left(\int_0^\infty t^{(s-\alpha)/p}f(t) \left[\frac{u}{t}\right]^{\alpha/p} dt\right)^{1/r} \cdot \left(\int_0^\infty t^{\alpha/p}g\left(\frac{u}{t}\right) dt\right)^{1/\alpha'}.$$

The last integral is $u^{-\alpha/p'} \|g\|_{p,\alpha'}$, and the middle term is $\|f\|_{p,s}$. Now raise both sides to the $r$th power, multiply by $u^{r/p - 1}$ and integrate with respect to $u$. We obtain

$$\|h\|_{p,r}^r \leq \left(\int_0^\infty t^{(s-\alpha)/p}f(t) \int_0^\infty u^{r/p - 1 - \alpha/p}g\left(\frac{u}{t}\right) \frac{dt}{u}\right) \cdot \|g\|_{p,\alpha'} \cdot \|f\|_{p,s}.$$

The integral in $u$, since $1 - \alpha/s' = \alpha/r$, becomes

$$\int_0^\infty u^{(s-\alpha)/p}g\left(\frac{u}{t}\right) \frac{du}{u} = t^{\alpha/p} \cdot \|g\|_{p,\alpha'},$$

and hence the first factor is

$$\int_0^\infty t^{(s-\alpha)/p} \cdot t^{\alpha/p}f(t) \frac{dt}{t} = \|f\|_{p,s}.$$

If we extract the $r$th root we obtain the desired norm inequality.

Remark. This is a “Young’s type” convolution inequality applied to the second index in the $L(p, q)$-spaces and our proof is a suitable adaptation of the usual proof for convolutions [7, p. 37].

The next lemma is needed for the integrals of the type (1), (2) for the maximal function.

Lemma 4. For $f, g: \mathbb{R}^+ \to \mathbb{R}^+$, let $F(u) = \int_0^\infty f(t)g(tu) dt$. If $1 < p < \infty$, $1/s + 1/\alpha \geq 1$, $1/r = 1/s + 1/\alpha - 1$, then

$$\|F\|_{p,r} \leq \|f\|_{p,s} \|g\|_{p,\alpha}.$$

Proof. If we let $t = 1/\tau$, then

$$F(u) = \int_0^\infty f(1/\tau)g(u\tau) \frac{d\tau}{\tau} \equiv \int_0^\infty f_1(\tau)g(\frac{u}{\tau}) \frac{d\tau}{\tau}.$$

Thus by Lemma 1, $\|F\|_{p,r} = \|f_1\|_{p,s} \|g\|_{p,\alpha}$, and

$$\|f_1\|_{p,s}^s = \int_0^\infty \left[\tau^{1/p-1}f\left(\frac{1}{\tau}\right)\right]^s \frac{d\tau}{\tau} = \int_0^\infty \tau^{s-1/p-1}f\left(\frac{1}{\tau}\right)^s d\tau = \int_0^\infty t^{s-1/p}f(t)^s \frac{dt}{t} = \|f\|_{p,s}^s.$$
6. We apply now Lemmas 3 and 4 to our original problem of finding a \( w \) between \( u, v \) so that \((u, w) \in A_p, (w, v) \in A_p\).

First we state Lemma 4 in the context of \((Mf)^*_p(\xi) \leq A \int_0^\infty \Phi(t)f^*_p(t\xi) \, dt\).

**Theorem 5.** If \( 1/s + 1/\sigma \geq 1, 1/r = 1/s + 1/\sigma - 1 \), then \( \|Mf\|_{p, r, \mu} \leq c\|\Phi\|_{p, s, \tau} \|f\|_{p, \sigma, \nu}. \)

**Theorem 6.** Let \((u, v)\) be a pair of functions for which there is a function \( w \) such that \( \Phi_1 = \Phi_{u, w} \in L(\rho, \sigma), \Phi_2 = \Phi_{w, v} \in L(\rho, \sigma) \), \( 1/s + 1/\sigma + 1/\rho \geq 2 \), and \( 1/\tau = 1/s + 1/\sigma + 1/\rho - 2 \). Then

\[
\|M_2f\|_{p, r, \mu} \leq c\|\Phi_1\|_{p, s, \tau} \|\Phi_2\|_{p, \sigma, \nu} \|f\|_{p, \rho, r}.
\]

**Proof.** We have seen that

\[
(M_2f)^*_p(\xi) \leq c\int_0^\infty \Phi_1(t)\Phi_2(u/t) \, dt/t.
\]

If \( 1/r = 1/s + 1/\sigma - 1 \), then by Lemma 3,

\[
\|\Psi\|_{p, r} \leq \|\Phi_1\|_{p, s} \|\Phi_2\|_{p, \sigma}.
\]

Finally, since \( 1/\tau = 1/r + 1/\rho - 1 \), Lemma 4 gives \( \|M_2f\|_{p, r, \mu} \leq c\|\Psi\|_{p, r} \|f\|_{p, \rho, r} \) and the proof is complete.

**References**


2. Lennart Carleson and Peter W. Jones, Weighted norm inequalities and a theorem of Koosis, preprint.


