AN ALTERNATE VARIATIONAL PRINCIPLE
FOR $\Delta u - u + |u|^{r-1}\text{sgn } u = 0$

CHARLES V. COFFMAN

ABSTRACT. An alternate variational principle for the equation in the title has been proposed by H. A. Levine. We analyse the relation between this principle and the Rayleigh quotient that has been used previously for the variational study of this problem in $\mathbb{R}^N$. The main result is an existence theorem for $W^{1,2}(\mathbb{R}^N)$-solutions of the variational problem posed by Levine.

1. Introduction. This note is concerned with the following differential equation in $\mathbb{R}^N (N \geq 2)$.

(1.1) $\Delta u - u + |u|^{r-1}\text{sgn } u = 0$.

and more generally with the problem.

(1.2) $\nabla (|\nabla u|^{p-1}\text{sgn } \nabla u) - |u|^{p-1}\text{sgn } u + |u|^{r-1}\text{sgn } u = 0,$

in $\mathbb{R}^N$. Here

$$\text{sgn } \nabla u = |\nabla u|^{-1} \nabla u \quad \text{if } \nabla u \neq 0, \quad \text{sgn } \nabla u = 0 \quad \text{otherwise.}$$

Variational studies of (1.1) have made use of the "Rayleigh quotient"

(1.3) $\left( \int (|\nabla u|^2 + u^2) \, dx \right)^{r/2} / \int |u|^r \, dx$

(here $\int (\cdot) \, dx$ denotes integration over $\mathbb{R}^N$), see e.g. [1]. In his abstract [3], H. A. Levine discussed the case $r = 5, N = 2$ of (1.1) and proposed for the variational study of this equation the quotient

(1.4) $\int |\nabla u|^2 \, dx \left( \int u^2 \, dx \right)^{1/2} / \left( \int u^s \, dx \right)^{1/2}$;

note that the latter quotient is not only homogeneous in $u$, like (1.3), but also is invariant under change of scale in $\mathbb{R}^2$. Since the preparation of the original version of this note the details of Levine's work have appeared in [4].

In what follows we shall denote, when $p, r, N$ are assumed to be given,

(1.5) $J_1(u) = \frac{\left( \int (|\nabla u|^p + |u|^p) \, dx \right)^{1/p}}{\left( \int |u|^r \, dx \right)^{1/r}}$;

Received by the editors January 9, 1980 and, in revised form, July 6, 1982.
1980 Mathematics Subject Classification. Primary 35J20.
Key words and phrases. Variational principle, nonlinear elliptic boundary value problem.
\textsuperscript{1}This work was supported by NSF Grant MCS 71-02776 A05.
note that the infimum of $J_1(u)$ over $C_0^\infty(R^N)\setminus\{0\}$ is $K_{p,r}^{-1}$, where $K_{p,r}$ denotes the norm of the imbedding $W^{1,p}(R^N) \subset L^r(R^N)$. (Our notation for Sobolev spaces follows that of [2].) For the appropriate values of $p$, $r$, $N$ we define

$$J_0(u) = \left( \frac{\int |\nabla u|^p \, dx}{\left( \int |u|^r \, dx \right)^{1/r}} \right)^{\rho / \beta} \left( \int |u|^p \, dx \right)^{\beta / \rho},$$

where $\alpha$ and $\beta$ are positive and are uniquely determined by the requirements that $J_0$ be homogeneous in $u$ and invariant under change of scale in $R^N$ (i.e. for any real positive scalars $\tau, \lambda$, if $w(x) = \lambda u(\tau x)$ then $J_0(w) = J_0(u)$). Indeed one easily sees that if $p > 1$ then $J_0$ can be formed in accordance with these requirements if and only if

$$0 < \frac{1}{p} - \frac{1}{r} < \frac{1}{N},$$

and that $\alpha, \beta$ are given by

$$\alpha = \left( \frac{1}{p} - \frac{1}{r} \right) N, \quad \beta = 1 - \alpha.$$

The pair of positive numbers $(p, r)$ will be said to be admissible for dimension $N$ if $p > 1$ and (1.7) holds.

Here we analyse the relation between these two variational principles. The interesting feature of the quotient $J_0$ is that (assuming $p > 1$) it can be formed in accordance with the requirements set down above precisely when the variational problem

$$J_1(u) = \min_{u \in W^{1,p}(R^N)},$$

has a solution. One outcome of this analysis is a new necessary condition for a "ground state" solution of (1.2), namely (1.11). Our result is the following

**Theorem.** Let $(p, r)$ be admissible for dimension $N$. Then there exists a nonnegative radially symmetric function $u \in W^{1,p}(R^N)\setminus\{0\}$ which minimizes the quotient $J_1$ over $W^{1,p}(R^N)\setminus\{0\}$. If $u$ is any minimizer of $J_1$ satisfying the normalization

$$\int (|\nabla u|^p + |u|^p) \, dx = \int |u|^r \, dx$$

then $u$ is a weak solution of (1.2), i.e.,

$$\int \left( |\nabla u|^{p-1} (\text{sgn} \, u) \cdot \nabla v + |u|^{p-1} (\text{sgn} \, u) v \right) \, dx = \int |u|^{r-1} (\text{sgn} \, u) v \, dx,$$

for all $v \in W^{1,p}(R^N)$.

A function $u \in W^{1,p}(R^N)\setminus\{0\}$ minimizes $J_1$ over $W^{1,p}(R^N)\setminus\{0\}$ if and only if it satisfies

$$\alpha \int |u|^p \, dx = \beta \int |\nabla u|^p \, dx,$$

and minimizes $J_0$ over the same set.
The author is grateful to the referee for having suggested some simplifications, both in proofs and in presentation.

2. Proof of the Theorem. We begin by proving the final assertion. For \( u \in W^{1,p}(\mathbb{R}^N) \setminus \{0\} \) let \( a = \beta f \| u \|^p \) and \( b = \alpha f \| u \|^p \). Then since \( a \approx b \leq \alpha a + \beta b \), with equality if and only if \( a = b \), we have that

\[
J_0(u) \leq \left[ \alpha^{\frac{1}{p}} \beta^{rac{1}{p}} \right] J_1(u)
\]

with equality if and only if (1.11) holds. For an appropriate choice of \( \lambda \), the function \( w \), given by \( w(x) = u(\lambda x) \) will satisfy (1.11) while \( J_0(w) = J_0(u) \). The assertion follows immediately.

In what follows we use the notation

\[
\| u \|_{1,p} = \left( \int (|\nabla u|^p + |u|^p) \right)^{1/p}
\]

for the norm on \( W^{1,p}(\mathbb{R}^N) \); \( V_p(\mathbb{R}^N) \) denotes the subspace of \( W^{1,p}(\mathbb{R}^N) \) that consists of all radially symmetric functions.

**Lemma 2.1.** Let \( 1 < p < r < \infty \). There exist positive constants \( C, \gamma \) such that for \( u \in V_p(\mathbb{R}^N) \) and \( \rho_0 > 0 \)

\[
\int_{|x| > \rho_0} |u|^r \, dx \leq C \rho_0^{-\gamma} \| u \|_{1,p}^r.
\]

**Proof.** For \( u \) as indicated we have, by Hölder’s inequality,

\[
|u(x)|^p \leq p \int_{|x|}^{\infty} |u|^{p-1} \left| \frac{\partial u}{\partial \rho} \right| \, d\rho
\]

\[
\leq \left( \frac{p}{q} \right)^{1/q} \left( \int_{|x|}^{\infty} |u|^p \, d\rho \right)^{1/q} \left( \int_{|x|}^{\infty} \left\| \frac{\partial u}{\partial \rho} \right\|^p \, d\rho \right)^{1/p}
\]

\[
\leq \left( \frac{p}{q} \right)^{1/q} \rho_0^{-\gamma} \left( \left\| \frac{\partial u}{\partial \rho} \right\|^p + |u|^p \right) \, d\rho
\]

and hence

\[
|u(x)|^p \leq \text{const} \cdot |x|^{-(N-1)} \| u \|_{1,p}^p.
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \). If

\[
r > pN/(N-1),
\]

then (2.1), with \( \gamma = (N-1)r/p - N \), follows directly from (2.2). If (2.3) fails to hold then we choose \( \rho_0 > pN/(N-1) \) and use the interpolation

\[
\int_{|x| > \rho_0} |u|^r \, dx \leq \left( \int |u|^p \, dx \right)^{(r_0-r)/(r_0-p)} \left( \int_{|x| > \rho_0} |u|^{r_0} \, dx \right)^{(r-p)/(r_0-p)}
\]

to obtain the asserted inequality.

**Remark.** The inequality (2.1) and its significance in regard to the problem at hand were suggested by Nehari’s paper [5].
**Lemma 2.2.** Let the pair \((p, r)\) be admissible for dimension \(N\), then the space \(V^p(R^N)\) embeds compactly in \(L'(R^N)\).

**Proof.** Since \(V^p(R^N)\) is reflexive, it suffices to show that a sequence \(\{w_n\}\) that converges weakly to 0 in \(V^p(R^N)\) converges to 0 in norm in \(L^p(R^N)\). This follows immediately from Lemma 2.1 and the compactness of the imbedding

\[ W^{1,p}(B_p) \subseteq L'(B_p) \]

where \(B_p = \{ x \in R^N : |x| \leq p \}\); cf. (1.7).

**Completion of the proof.** Since both \(W^{1,p}(R^N)\) and \(L(R^N)\) are lattices in which \(u\) and \(|u|\) have the same norm, replacement of \(u\) by \(|u|\) does not affect the value of \(J_1\). If the nonnegative function \(u \in W^{1,p}(R^N)\setminus\{0\}\) is replaced by its Schwarz symmetrization \(\tilde{u} \in V^p(R^N)\) (see [6] and the references there) then

\[ \int |\tilde{u}|^p dx = \int |\tilde{u}|^p dx \text{ for } 1 \leq \sigma < \infty \text{ while } \int \nabla \tilde{u} \nabla \tilde{u} dx < \int \nabla u \nabla u dx \text{ unless } \tilde{u} \text{ is merely a translate of } u. \]

It follows that we can choose a minimizing sequence \(\{u_n\}\) for \(J_1\) in such a way that \(u_n\) is nonnegative and \(u_n \in V^p(R^N)\setminus\{0\}\) for \(n = 1, 2, \ldots\).

\[ \int |u_n|^p dx = 1, \quad n = 1, 2, \ldots; \]

and \(\{u_n\}\) is weakly convergent, say to \(u_0\), in \(V^p(R^N)\). It follows from Lemma 2.2 that \(\{u_n\}\) converges strongly to \(u_0\) in \(L'(R^N)\). Thus in view of (2.4)

\[ \int |u_0|^p dx = 1 \]

and in particular \(u_0 \neq 0\). In view of the weak lower semicontinuity of \(||| \cdots |||_{1,p}\) on \(W^{1,p}(R^N)\) it readily follows that \(u_0\) minimizes \(J_1\) over \(W^{1,p}(R^N)\setminus\{0\}\). Let \(u\) denote the function that results when \(u_0\) is normalized to satisfy (1.9). It follows by standard results in the calculus of variations that if \(u\) minimizes \(J_1\) and satisfies (1.9) then \(u\) must be a weak solution of (1.1).

3. **The case \(a = 1\).** In the limiting case in which the second inequality in (1.7) is replaced by equality we have obviously

\[ J_1(u) > J_0(u), \quad u \in W^{1,p}(R^N)\setminus\{0\}. \]

while if \(u_\lambda(x) = u(\lambda x)\) then \(J_0(u_\lambda)/J_1(u_\lambda) \to 1\) as \(\lambda \to \infty\). It follows that \(J_1\) cannot assume a minimum on \(W^{1,p}(R^N)\). The infimum of \(J_0\) however can be explicitly calculated in this case. [6]. For example, in the case \(p = 2\), the change of variable \(t = \ln \rho, w(t) = \rho^{2/\gamma-2}u(\rho)\), transforms the equation

\[ \frac{d^2u}{dt^2} + \frac{N-1}{\rho} \frac{du}{dp} + |u|^\gamma \text{sgn } u = 0. \]

to the autonomous equation

\[ \dot{w} + \left( N - \frac{2r}{r-2} \right) \dot{w} - \frac{2}{r-2} \left( N - \frac{2r-2}{r-2} \right) w + |w|^\gamma \text{sgn } w = 0. \]
When $\alpha = 1$, i.e. $N = 2r/(r - 2)$ then this reduces to

$$\ddot{w} + \left(\frac{2}{r - 2}\right)^2 w + |w|^{r-1}\text{sgn}\ w = 0,$$

which can be integrated. From the formulas found in [6] one sees that the minimizer of $J_0$ belongs to $W^{1,p}(\mathbb{R}^N)$ only when $N > p^2$.

REFERENCES