PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 87, Number 4, April 1983

$\aleph_0$-CATEGORICAL DISTRIBUTIVE LATTICES OF FINITE BREADTH

JAMES H. SCHMERL[1]

ABSTRACT. Every $\aleph_0$-categorical distributive lattice of finite breadth has a finitely axiomatizable theory. This result extends the analogous result for partially ordered sets of finite width.

0. Introduction. This note is mainly concerned with the following theorem.

THEOREM 1. *Every $\aleph_0$-categorical distributive lattice of finite breadth has a finitely axiomatizable theory.*

For general references on distributive lattice theory we suggest [**1** and **5**]. Relevant material concerning $\aleph_0$-categoricity can be found in [**8** and **9**]. The article [**7**] also considers $\aleph_0$-categorical distributive lattices, but in a slightly different vein than is done here.

Theorem 1 generalizes a corresponding result for partially ordered sets (Theorem 1 of [**9**]) that every $\aleph_0$-categorical poset of finite width has a finitely axiomatizable theory. In fact, this latter result, which is an important ingredient in the proof of Theorem 1, follows fairly immediately from Theorem 1 as will be demonstrated in §3 by appropriately interpreting the theory of posets of width n in the theory of distributive lattices of breadth n.

The notion of breadth of a lattice seems not to be a well discussed concept; it is not mentioned in either of the references [**1** or **5**], but is briefly referred to in [**2**]. Yet, for distributive lattices it becomes a particularly stable and transparent notion. Various equivalent characterizations will be presented in §1.

The proof of Theorem 1 will be presented in §2.

1. Breadth. A lattice $(A, \wedge, \vee)$ has *breadth* $\leqslant n$ iff whenever $a_0, a_1, \ldots, a_n \in A$ there is some $i \leqslant n$ such that $a_0 \vee a_1 \vee \cdots \vee a_n = a_0 \vee a_1 \vee \cdots \vee a_{i-1} \vee a_{i+1} \vee \cdots \vee a_n$. If one identifies (as we shall do) the lattice $(A, \wedge, \vee)$ with the poset $(A, \leqslant)$ (where $x \leqslant y$ iff $x = x \wedge y$ iff $y = x \vee y$), then it is immediate that breadth 1 lattices are merely chains. If $L_0, L_1, \ldots, L_{n-1}$ are chains, then $L_0 \times L_1 \times \cdots \times L_{n-1}$ is a distributive lattice which has breadth $\leqslant n$, and if each L_i is nontrivial (i.e. $|L_i| \geqslant 2$), then it has breadth precisely n. In particular, if each $|L_i| = 2$, then $B_n = L_0 \times \cdots \times L_{n-1}$ is the *Boolean lattice* with breadth n.

Received by the editors May 3, 1982 and, in revised form, August 23, 1982.
1980 *Mathematics Subject Classification*. Primary 03C35, 03C15; Secondary 06D05.
Key words and phrases. $\aleph_0$-categoricity, distributive lattice, finite axiomatizability.
[1]Partially supported by NSF Grant No. MCS-7905028-01.

0002-9939/82/0000-0871/$02.50

Suppose now that $\mathfrak{D} = (D, \wedge, \vee)$ is an arbitrary distributive lattice. Then $\langle \theta_0, \theta_1, \ldots, \theta_{n-1} \rangle$ is a *coordinatization* of $\mathfrak{D}$ if the following two conditions hold:

(1) each θ_i is a congruence relation of $\mathfrak{D}$ such that $\mathfrak{D}/\theta_i$ has breadth 1;

(2) $\theta_0 \cap \theta_1 \cap \cdots \cap \theta_{n-1}$ is the trivial congruence relation.

A partially ordered set $\mathfrak{B} = (B, \prec)$ has *width* $\leqslant n$ if $\mathfrak{B}$ has no antichains of length greater than n; and it has *dimension* $\leqslant n$ if there are chains $B_0, B_1, \ldots, B_{n-1}$ such that $\mathfrak{B}$ is isomorphic to a subposet of $B_0 \times B_1 \times \cdots \times B_{n-1}$. We mention here the fundamental theorem of Dilworth [3]: Every poset of width n can be covered by n chains.

We now state a series of various characterizations of breadth for distributive lattices.

PROPOSITION 2. *For a distributive lattice* $\mathfrak{D} = (D, \wedge, \vee)$ *and* $1 \leqslant n < \omega$ *the following are equivalent*:

(0) $\mathfrak{D}$ *has breadth* $\leqslant n$;

(1) $\mathfrak{D}$ *does not embed the Boolean lattice* B_{n+1};

(2) *the set of prime ideals of* $\mathfrak{D}$ *partially ordered by inclusion has width* $\leqslant n$;

(3) $\mathfrak{D}$ *has a coordinatization* $\langle \theta_0, \theta_1, \ldots, \theta_{n-1} \rangle$;

(4) *there are chains* $L_0, L_1, \ldots, L_{n-1}$ *such that* $L_0 \times L_1 \times \cdots \times L_{n-1}$ *embeds* $\mathfrak{D}$;

(5) $\mathfrak{D}$ (*considered as a poset*) *has dimension* $\leqslant n$.

All the above characterizations are quite standard. The proposition can be proved by the circle of implications

$$(0) \Rightarrow (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (0),$$

all of which are rather routine except possibly for (2) $\Rightarrow$ (3) which requires Dilworth's theorem and the Prime Ideal Theorem (PIT) for distributive lattices (Theorem 7.15 of [5] or Theorem III.4.1 of [1]). We shall prove as Theorem 5 a version of (2) $\Rightarrow$ (3) in an $\aleph_0$-categorical setting.

In the following theorem we present another, less standard, characterization of distributive lattices of breadth n. Not only the theorem but its proof will be used in the proof of Theorem 1.

Let $\mathfrak{B} = (B, \prec)$ be a poset of width n, and let $S(\mathfrak{B})$ be the set of antichains of $\mathfrak{B}$ of length n. For $X, Y \in S(\mathfrak{B})$ we make the following two definitions: $X \wedge Y$ is the set of minimal elements of $X \cup Y$, and $X \vee Y$ is the set of maximal elements of $X \cup Y$. Clearly, both $X \wedge Y$ and $X \vee Y$ are in $S(\mathfrak{B})$. With these definitions, $\mathfrak{S}(\mathfrak{B}) = (S(\mathfrak{B}), \wedge, \vee)$ becomes a lattice. In fact, it is a consequence of Dilworth's theorem that $\mathfrak{S}(\mathfrak{B})$ is a distributive lattice [4], which, as is easily seen, has breadth n. Conversely, there is the following theorem.

THEOREM 3. *If* $\mathfrak{D} = (D, \wedge, \vee)$ *is a distributive lattice of breadth* n, *then there is a poset* $\mathfrak{B} = (B, \prec)$ *of width* n *such that* $\mathfrak{D} \cong \mathfrak{S}(\mathfrak{B})$.

PROOF. Let $\langle \theta_0, \theta_1, \ldots, \theta_{n-1} \rangle$ be a coordinatization of $\mathfrak{D}$, which exists by Proposition 2(3). Then each $\mathfrak{D}/\theta_i$ is a lattice of breadth 1; that is, $\mathfrak{D}/\theta_i$ is a chain $(L_i, <_i)$.

Without loss of generality assume that $L_i \cap L_j = \varnothing$ whenever $i < j < n$. Let $B = L_0 \cup L_1 \cup \cdots \cup L_{n-1}$, and define $\prec$ on B so that if $a \in L_i$ and $b \in L_j$, then

$$a \prec b \Leftrightarrow \forall x \in D\big[x \leqslant_i a \to x <_j b\big].$$

It is easy to see that $\mathfrak{B} = (B, \prec)$ is a poset. Furthermore, each $(L_i, <_i)$ is a subposet of $\mathfrak{B}$, so that $\mathfrak{B}$ has width $\leqslant n$. In fact, $\mathfrak{B}$ has width n since $\{a/\theta_0, \ldots, a/\theta_{n-1}\}$ is an antichain of length n for each $a \in D$. It remains to demonstrate that $\mathfrak{D} \cong \mathfrak{S}(\mathfrak{B})$. The intended isomorphism should be clear: it is just the function $f\colon D \to S(\mathfrak{B})$, where

$$f(a) = \{a/\theta_0, a/\theta_1, \ldots, a/\theta_{n-1}\}.$$

It is clear that f is an embedding of $\mathfrak{D}$ into $\mathfrak{S}(\mathfrak{B})$, so we need only show that f is onto. This follows immediately from the following which we now prove:

(*) Suppose X is an antichain of $\mathfrak{B}$ (which is not necessarily maximal). Then there exists $a \in D$ such that $X \subseteq f(a)$.

The proof of (*) proceeds by induction on $|X|$. If $|X| = 1$, where $X = \{b\}$ for $b \in L_i$, then choose any $a \in b$. If $|X| = 2$, then without loss of generality suppose $X = \{b_0, b_1\}$, where $b_0 \in L_0$ and $b_1 \in L_1$. Since neither $b_0 \prec b_1$ nor $b_1 \prec b_0$, there are $x, y \in D$ such that $x \in b_0 \leqslant_0 y/\theta_0$ and $y \in b_1 \leqslant_1 x/\theta_1$. Then let $a = x \wedge y$. Finally, suppose $3 \leqslant |X| \leqslant n$, and without loss of generality suppose $b_0, b_1, b_2 \in X$ where $b_0 \in L_0$, $b_1 \in L_1$, $b_2 \in L_2$. By the inductive hypothesis there are a_0, a_1, $a_2 \in D$ such that if $j \in \{0, 1, 2\}$, then $x - \{b_j\} \subseteq f(a_j)$. Then, say, $b_0 \leqslant_0 a_0/\theta_0$ and $b_1 \leqslant_1 a_1/\theta_1$, so that $a = a_0 \wedge a_1$ will be as required. □

2. The proof of Theorem 1. This section is devoted to the proof of Theorem 1. We will make use of the following $\aleph_0$-categorical version of Dilworth's theorem, which extends Theorem 6.2 of [**8**]. A proof of Theorem 4 is implicit in the proofs of Theorem 6.2 of [**8**] and Theorem 1 of [**9**]. As was pointed out by M. Parigot, there is a slight gap occurring in the final paragraph of the exposition of the proof of Theorem 6.2 of [**8**], so we take this opportunity to fill that gap by presenting a more detailed version of the proof of Theorem 4.

THEOREM 4. *If $(\mathfrak{B}, \prec)$ is a countable and $\aleph_0$-categorical structure, where $(B, \prec)$ is a poset of width n, then there are chains $C_0, C_1, \ldots, C_{n-1}$ whose union is B such that:*

(1) *$(\mathfrak{B}, \prec, C_0, C_1, \ldots, C_{n-1})$ is $\aleph_0$-categorical;*

(2) *if $(\mathfrak{B}, \prec, C_0, C_1, \ldots, C_{n-1})$ has a finitely axiomatizable theory, then so does $(\mathfrak{B}, \prec)$.*

PROOF. We will rely on results from both [**8** and **9**] for this proof. The proof will proceed by induction on n, the width of the poset $(B, \prec)$. In order to maintain the induction, we will require that the chains $C_0, C_1, \ldots, C_{n-1}$ satisfy not only conditions (1) and (2), but also condition:

(3) if E is a simple component of $(B, \prec)$ which has width k, then $E \subseteq C_0 \cup C_1 \cup \cdots \cup C_{k-1}$.

If $n = 1$, the result is trivial.

For the inductive step, suppose $(B, \prec)$ has width $n \geqslant 2$. Let N be the natural refinement of the simple splitting of $(B, \prec)$. (Refer to §0 of [9].) Let r be the least number such that each simple component is partitioned into no more than r N-components. Using Proposition 1.4 and Corollary 1.3 of [9], we can obtain a partition $\{B_0, B_1, \ldots, B_{r-1}\}$ of B such that:

(4) $(\mathfrak{B}, \prec, B_0, B_1, \ldots, B_{r-1}\}$ is $\aleph_0$-categorical;

(5) if $(\mathfrak{B}, \prec, B_0, B_1, \ldots, B_{r-1}\}$ has a finitely axiomatizable theory, then so does $(\mathfrak{B}, \prec)$;

(6) if E is a simple component of $(B, \prec)$ which has m N-components, then $\{E \cap B_0, E \cap B_1, \ldots, E \cap B_{m-1}\}$ is just the set of its N-components.

Now define $<$ on B so that if $a, b \in B$, then $a < b$ iff one of the following holds:

(7) a and b are in distinct simple components and $a \prec b$;

(8) a and b are in the same N-component and $a \prec b$;

(9) a and b are in the same simple component, and $a \in B_i$, $b \in B_j$ for some $i < j < r$.

Let $\mathfrak{B}' = (\mathfrak{B}, \prec, B_0, B_1, \ldots, B_{r-1})$, and notice that $<$ is definable in $\mathfrak{B}'$. Clearly, $(B, <)$ is a poset, and by Lemma 6.4 of [8], it has width $< n$; say the width is m. Apply the inductive hypothesis to the structure $(\mathfrak{B}', <)$, obtaining chains $D_0, D_1, \ldots, D_{m-1}$.

For each simple component E of $(B, \prec)$, the width of E being r, consider the set $\mathcal{C}_E = \{X \cap D_j\colon X$ is an N-component of E having width $> j\}$. Partially order $\mathcal{C}_E$ by $<$ so that if $X, Y \in \mathcal{C}_E$, then $X < Y$ iff $x < y$ for each $x \in X$ and $y \in Y$. By Lemma 6.6 of [8], $(\mathcal{C}_E, <)$ has width $\leqslant r$ so by Dilworth's theorem (for finite posets only), it can be covered by r chains. So, by again applying Proposition 1.4 of [8] (not to $(\mathfrak{B}', \prec)$ directly, but to an appropriate definitional extension of it), there are chains $C_0, C_1, \ldots, C_{n-1}$ whose union is B such that the structure $((\mathfrak{B}, <, B_0, \ldots, B_{r-1}), \prec, C_0, C_1, \ldots, C_{n-1})$ satisfies (1)–(3). But then $(\mathfrak{B}, \prec, C_0, \ldots, C_{n-1})$ satisfies (1) and (3), and by (5) it also satisfies (2). □

REMARK. In Theorem 4 we can also require of the chains that each of them be maximal.

Theorem 4 will be generalized in the following manner.

THEOREM 5. *If $(\mathfrak{P}, \wedge, \vee)$ is a countable and $\aleph_0$-categorical structure, where $(D, \wedge, \vee)$ is a distributive lattice of breadth n, then there is a coordinatization $\langle \theta_0, \theta_1, \ldots, \theta_{n-1} \rangle$ of $(D, \wedge, \vee)$ such that:*

(1) *$(\mathfrak{P}, \wedge, \vee, \theta_0, \theta_1, \ldots, \theta_{n-1})$ is $\aleph_0$-categorical;*

(2) *if $(\mathfrak{P}, \wedge, \vee, \theta_0, \theta_1, \ldots, \theta_{n-1})$ has a finitely axiomatizable theory, then so does $(\mathfrak{P}, \wedge, \vee)$.*

PROOF. Let $\mathfrak{P}$ be the set of prime ideals of $(D, \wedge, \vee)$ which, by Proposition 2(2), has width n when ordered by inclusion. For each $a \in D$ let $\mathfrak{P}_a = \{I \in \mathfrak{P}\colon a \in I\}$, and let $\mathfrak{P}_a^*$ be the set of those prime ideals which are minimal in $\mathfrak{P}_a$. The PIT implies that for each $b \in D$, if it is not the case that $b \leqslant a$, then there exists $I \in \mathfrak{P}_a^*$ such that $b \notin I$.

Let $\psi(x, y)$ be a formula (in the language of lattice theory) asserting: $x \leqslant y$ and there is a unique $I \in \mathcal{P}_x^*$ such that $y \in I$. If $a, b \in D$ are such that $(D, \wedge, \vee) \models \psi(a, b)$, then let I_{ab} be that unique prime ideal in $\mathcal{P}_a^*$ containing b. Notice that $\mathcal{P}_a^* = \{I_{ab}: (D, \wedge, \vee) \models \psi(a, b)\}$. Now let $\phi(x, y, z)$ be a formula asserting $\psi(x, y)$ and $z \in I_{xy}$. The existence of the formulas $\psi(x, y)$ and $\phi(x, y, z)$ is guaranteed by Ryll-Nardzewski's Theorem which implies that any relation on $(D, \wedge, \vee)$ invariant under all automorphisms of $(D, \wedge, \vee)$ is definable by a first-order formula.

Let $\chi(x_1, y_1, x_2, y_2)$ be the formula

$$\psi(x_1, y_1) \wedge \psi(x_2, y_2) \wedge [\forall z(\phi(x_1, y_1, z) \rightarrow \phi(x_2, y_2, z))],$$

which defines a partial ordering on the partition of $A = \{(a, b) \in D^2: (D, \wedge, \vee) \models \psi(a, b)\}$ defined by the formula

$$\psi(x_1, y_1) \wedge \psi(x_2, y_2) \wedge [\forall z(\psi(x_1, y_1, z) \leftrightarrow \psi(x_2, y_2, z))].$$

This partial ordering is essentially a subordering of $(\mathcal{P}, \subseteq)$, and thus has width $\leqslant n$. By Theorem 4 there are maximal χ-chains $C_0, C_1, \ldots, C_{n-1} \subseteq A$ whose union is A such that:

(i) $(\mathfrak{D}, \wedge, \vee, C_0, C_1, \ldots, C_{n-1})$ is $\aleph_0$-categorical;

(ii) if $(\mathfrak{D}, \wedge, \vee, C_0, C_1, \ldots, C_{n-1})$ has a finitely axiomatizable theory, then so does $(\mathfrak{D}, \wedge, \vee)$.

(Precisely, Theorem 4 is not being applied to the structure $(\mathfrak{D}, \wedge, \vee)$, but to another structure which is definitionally equivalent to it which codes pairs from D by single points.)

For each $i < n$ we make the following definition

$$\theta_i(w, z) \equiv \forall xy[C_i(x, y) \rightarrow (\phi(x, y, z) \leftrightarrow \phi(x, y, w))],$$

thereby obtaining a coordinatization $\langle \theta_0, \theta_1, \ldots, \theta_{n-1} \rangle$ of $(D, \wedge, \vee)$. Clearly, each $(D, \wedge, \vee)/\theta_i$ has breadth 1. To see that $\theta_0 \cap \theta_1 \cap \cdots \cap \theta_{n-1}$ is the trivial congruence, consider elements $a \neq b$ of D. Without loss of generality, assume that not $b \leqslant a$, so by the PIT there is $I \in \mathcal{P}_a$ such that $b \notin I$. But then there is $I \in \mathcal{P}_a^*$ such that $b \notin I$. There is $c \in I$ such that $I = I_{ac}$ which can be obtained as follows: Let $J_0, J_1, \ldots, J_{k-1}$ be those ideals in $\mathcal{P}_a^*$ different from I, choose arbitrary $c_i \in I - J_i$, and then set $c = a \vee c_0 \vee c_1 \vee \cdots \vee c_{k-1}$. Thus, $(D, \wedge, \vee) \models \psi(a, c)$. Let $i < n$ be such that $(a, c) \in C_i$. But then $(D, \wedge, \vee) \models \phi(a, c, a) \wedge \neg\phi(a, c, b)$, so that $\neg\theta_i(a, b)$. Therefore, $\theta_0 \cap \theta_1 \cap \cdots \cap \theta_{n-1}$ is trivial. To complete the proof we need to verify conditions (1) and (2) of the theorem. Condition (1) easily follows from (i). To prove (2), it suffices, because of (ii), to show that each of $C_0, C_1, \ldots, C_{n-1}$ is definable in $(\mathfrak{D}, \wedge, \vee, \theta_0, \theta_1, \ldots, \theta_{n-1})$. It is easy to check that the definition

$$C_i(x, y) \equiv \psi(x, y) \wedge \forall zw[\phi(x, y, z) \wedge \theta_i(w, z) \rightarrow \phi(x, y, w)]$$

is adequate. $\square$

We are now prepared to complete the proof of Theorem 1. Let $\mathfrak{D} = (D, \wedge, \vee)$ be an $\aleph_0$-categorical distributive lattice of breadth n, and, without loss of generality, assume it is countable. Let $\langle \theta_0, \theta_1, \ldots, \theta_{n-1} \rangle$ be a coordinatization of $\mathfrak{D}$ as in Theorem 5.

Let $\mathfrak{B} = (B, \prec)$ be a poset of width n such that $\mathfrak{D} = \mathfrak{S}(\mathfrak{B})$; in fact, let $\mathfrak{B}$ be the actual poset constructed in the proof of Theorem 3 using the coordinatization $\langle \theta_0, \theta_1, \ldots, \theta_{n-1} \rangle$. Let $L_0, L_1, \ldots, L_{n-1}$ be the chains from that proof, and let f: $D \to S(\mathfrak{B})$ be the isomorphism defined there.

From the definable manner that $(B, \prec, L_0, L_1, \ldots, L_{n-1})$ was obtained from $(\mathfrak{D}, \theta_0, \theta_1, \ldots, \theta_{n-1})$, it is clear that $(B, \prec, L_0, L_1, \ldots, L_{n-1})$ is $\aleph_0$-categorical since $(\mathfrak{D}, \theta_0, \theta_1, \ldots, \theta_{n-1})$ is. The structure $(B, \prec, L_0, L_1, \ldots, L_{n-1})$ has a finitely axiomatizable theory by Theorem 1.1 of [**9**]. Since the function f is definable, it is clear that the structure $(\mathfrak{D}, \theta_0, \theta_1, \ldots, \theta_{n-1})$ is definable from $(B, \prec, L_0, L_1, \ldots, L_{n-1})$. Hence $(\mathfrak{D}, \theta_0, \theta_1, \ldots, \theta_{n-1})$ has a finitely axiomatizable theory. But then Theorem 5 implies that $\mathfrak{D}$ has a finitely axiomatizable theory, completing the proof of Theorem 1.

The same proof will work to prove the following slightly stronger version of Theorem 1.

THEOREM 6. *Let $\mathfrak{D} = (D, \wedge, \vee, D_0, D_1, \ldots, D_{n-1})$ be $\aleph_0$-categorical, where $(D, \wedge, \vee)$ is a distributive lattice of finite breadth and each D_i is a sublattice. Then $\mathfrak{D}$ has a finitely axiomatizable theory.*

3. Interpretability. We shall demonstrate in this section that the theory of posets of width n is interpretable in the theory of distributive lattices of breadth n. This interpretation will be of a strong sort which we called in [**10**] a completely faithful interpretation. It is not really important here what the precise definition is. Suffice it to say that given Theorem 1 and the specific interpretation of Theorem 7, we will get as a consequence the finite axiomatizability of the theory of any $\aleph_0$-categorical poset of finite width. It will incidentally also yield the undecidability of the theory of distributive lattices of breadth 2 from the corresponding theorem for posets of width 2.

THEOREM 7. *The theory of posets of width n is completely faithfully interpretable in the theory of distributive lattices of breadth n.*

PROOF. We begin by giving a semantic description of the interpretation π of the theory of posets of width n in the theory of distributive lattices of breadth n.

Let $\mathfrak{D} = (D, \wedge, \vee)$ be an arbitrary distributive lattice of breadth n. Let $K(\mathfrak{D})$ be the set of those elements in D which have a unique maximum predecessor; thus,

$$a \in K(\mathfrak{D}) \Leftrightarrow \mathfrak{D} \models \exists y \forall z (z \leqslant y \Leftrightarrow z < a).$$

Each element in $K(\mathfrak{D})$ is certainly a join-irreducible, so that by Proposition 2(2), the poset $(K(\mathfrak{D}), <)$ has width $\leqslant n$. Let $\mathfrak{D}^\pi = (K(\mathfrak{D}), <)$.

To see that this interpretation is completely faithful, let $\mathfrak{B} = (B, \prec)$ be an arbitrary poset of width n. A subset $X \subseteq B$ is *hereditary* if $x \in X$ and $y \prec x$ implies $y \in X$. Let $H(\mathfrak{B})$ be the set of hereditary subsets of B. Obviously, $H(\mathfrak{B})$ is closed under intersection and union so that $\mathfrak{H}(\mathfrak{B}) = (H(\mathfrak{B}), \cap, \cup)$ is a distributive lattice, which can be shown (using, for example, Proposition 2(2)) to have breadth n. Let

$\mathfrak{L}(\mathfrak{B}) = (L(\mathfrak{B}), \cap, \cup)$ be the sublattice of $\mathfrak{K}(\mathfrak{B})$ generated by all hereditary sets of the form

$$H_b = \{x \in B: x \preccurlyeq b\} \quad \text{and} \quad H^b = \{x \in B: \neg x \succcurlyeq b\}$$

for $b \in B$. To complete the proof it suffices to demonstrate the following easy

Claim. $K(\mathfrak{L}(\mathfrak{B})) = \{H_b: b \in B\}$.

Clearly $H_b \in K(\mathfrak{L}(\mathfrak{B}))$, for $Y = \{y \in B: y \prec b\} = H_b \cap H^b$ is the unique maximum predecessor of H_b.

Conversely, suppose that $X \in K(\mathfrak{L}(\mathfrak{B}))$ and that Y is the maximum predecessor of X. Let $b \in X - Y$ so that not $H_b \subseteq Y$. But then $H_b = X$ by the maximality of Y. □

COROLLARY 8. *The theory of distributive lattices of breadth* 2 *is undecidable.*

PROOF. The proof follows from Theorem 7 and the undecidability of the theory of posets of width 2 (Lemma 8.6 of [**8**]). □

The corollary extends the result [**6**] on the undecidability of theory of 2-dimensional posets.

REFERENCES

1. R. Balbes and P. Dwinger, *Distributive lattices*, Univ. of Missouri Press, Columbia, 1974.

2. G. Birkhoff, *Lattice theory*, 3rd ed., Amer. Math. Soc. Colloq. Publ., vol. 25, Amer. Math. Soc., Providence, R. I., 1967.

3. R. P. Dilworth, *A decomposition theorem for partially ordered sets*, Ann. of Math. (2) **51** (1950), 161–166.

4. _____, *Some combinatorial problems on partially ordered sets*, Combinatorial Analysis (R. Bellman and M. Hall, Jr., eds.), Proc. Sympos. Appl. Math., vol. 10, Amer. Math. Soc., Providence, R. I., 1960, pp. 85–90.

5. G. Grätzer, *Lattice theory: first concepts and distributive lattices*, Freeman, San Francisco, Calif., 1971.

6. A. B. Manaster and J. G. Rosenstein, *Two-dimensional partial orderings: undecidability*, J. Symbolic Logic **45** (1980), 133–143.

7. P. Olin, *Aleph-zero categorical Stone algebras*, J. Austral. Math. Soc. Ser. A **26** (1978), 337–347.

8. J. H. Schmerl, *Decidability and $\aleph_0$-categoricity of theories of partially ordered sets*, J. Symbolic Logic **45** (1980), 585–611.

9. _____, *Decidability and finite axiomatizability of theories of $\aleph_0$-categorical partially ordered sets*, J. Symbolic Logic **46** (1981), 101–120.

10. _____, *Arborescent structures*, II: *Interpretability in the theory of trees*, Trans. Amer. Math. Soc. **266** (1981), 629–643.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONNECTICUT, STORRS, CONNECTICUT 06268