ANOTHER INTERESTING PROPERTY CONCERNING THE PROBABILITY MEASURES ON THE RATIONALS

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Abstract. Let $X$ be a perfect, complete, separable metric space and $P(X)$ denote the space of Borel probability measures on $X$ equipped with the topology of weak convergence. If $Y$ is a countable dense subset of $X$ then $P(Y)$ is not a $G_{\delta_0}$ subset of $P(X)$. Furthermore if $X$ is separable, complete and metric, and $Y \subseteq X$, and $P(Y)$ is a $G_{\delta_0}$ subset of $P(X)$, then $P(Y)$ is in fact a $G_{\delta}$ subset of $P(X)$.

I. Introduction. Let $X$ be a separable metric space and $P(X)$ denote the space of Borel probability measures on $X$. The topology on $P(X)$ is the topology of weak convergence, i.e. $\lim m_1 = m$ iff $\limsup m_1(F) \leq m(F)$ for all closed $F \subseteq X$. Equivalent versions of the topology and other properties of $P(X)$ can be found in [4, pp. 39–50].

In [5] Preiss showed that if $Y$ is taken to be the rationals (or more generally is first category in itself) then $Y$ is not Prohorov. Preiss then used this result to show that if $Y$ is any separable metric coanalytic space, then $Y$ is Prohorov implies that $Y$ is topologically complete (which is equivalent to $P(Y)$ being topologically complete).

In this paper it is shown that if $Y$ is a countable dense subset of a perfect Polish space $X$, then $P(Y)$ is not a $G_{\delta_0}$ subset of $P(X)$. Then, using Theorem 2 of [5], it is shown that if $Y \subseteq X$ (Polish) and $P(Y)$ is a $G_{\delta_0}$ subset of $P(X)$, then $P(Y)$ is in fact a $G_{\delta}$ subset of $P(X)$.

This result is of the same flavor as that of Luther in [3] in which he showed that local compactness of $P(Y)$ (actually the existence of a compact neighborhood) implies that $P(Y)$ is compact. It is not difficult using Luther's result to show if $P(Y)$ is $\sigma$-compact then $P(Y)$ is compact.

These three results can be summarized as follows: if $Y$ is separable and metric then:

(1) if $Y$ is $\mathcal{C}4$ and Prohorov then $P(Y)$ is absolute $G_{\delta}$,
(2) if $P(Y)$ is absolute $G_{\delta_0}$ then $P(Y)$ is absolute $G_{\delta}$,
(3) if $P(Y)$ is absolute $F_\sigma$ then $P(Y)$ is absolute closed.

The necessary facts concerning absolute Borel class can be found in [2, p. 339].

II. Background. Two background theorems require explicit mention. The first is due to Prohorov and may be found in [4, p. 46, Theorem 6.5].

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1The results contained in this paper were obtained by the author while he was a student of Jack B. Brown at Auburn University.
Theorem A (Prohorov). If X is separable and metric, then P(X) is topologically complete iff X is.

The second may be found in [6, p. 430].

Theorem B. If X is complete, then F ⊆ X is topologically complete iff F is a $G_δ$ subset of X.

In [7] Varadarajan notes that a basis for the topology of weak convergence is formed by finite intersections of $\{m \in P(X): m(U) > m_0(U) - \varepsilon, i = 1, n\}$ where $m_0$ is some fixed probability measure, $U$ is open in $X$ and $\varepsilon > 0$. For our purposes we need to show that sets of a more restrictive form constitute a basis for $P(X)$.

Lemma 1. Sets of the form $N^*(m_0; \varepsilon; U_1, \ldots, U_n) = \{m \in P(X): m(U_i) > m_0(U_i) - \varepsilon, i = 1, n\}$ where $\varepsilon > 0$ and $U_1, \ldots, U_n$ are disjoint open subsets of $X$ such that $\Sigma_{i=1}^n m_0(U_i) = 1$, form a basis for the weak topology on $P(X)$.

Proof. The usual basis for the weak topology on $P(X)$ is defined by sets of the form $N(m_0; \varepsilon; f_1, \ldots, f_n) = \{m \in P(X): |\int f_i dm - \int f_i dm_0| < \varepsilon, j = 1, \ldots, n\}$ where the functions $f_j$ are required to be bounded continuous functions from $X$ into the reals (see [4]). For convenience assume that the $f_j$ are into $[0, 1]$.

As $N(m_0; \varepsilon; f_1, \ldots, f_n) = \bigcap_{i=1}^n N(m_0; \varepsilon; f_i)$, we need only consider $N(m_0; \varepsilon; f_i)$ and show that intersections may be accommodated.

Consider $N(m_0; \varepsilon; f)$ where $f$ is taken to be from $X$ into $[0, 1]$. Pick an integer $n > 2$ so that $(1/n) < (\varepsilon/8)$. Pick numbers $0 < a_0 < a_1 < \cdots < a_{n+1} = 1$ so that the mesh of the subdivision is less than $(1/n)$ and $m_0(\{f \in \{a_0, \ldots, a_n\}\}) = 0$. Note that the last property can be obtained since $n + 1$ points are chosen and only countably many points are the image under $f$ of sets of positive $m_0$-measure.

Let $U_0 = \{f \in [0, a_0]\}$, $U_i = \{f \in (a_{i-1}, a_i]\}$ for $1 \leq i \leq n$ and $U_{n+1} = \{f \in (a_n, 1]\}$. Note $\Sigma_{i=1}^{n+1} m_0(U_i) = 1$. Pick $0 < \delta < 1$ so that $\delta(n + 1) < (\varepsilon/8)$.

Consider $m \in N^*(m_0; (\delta/2^{n+1}); U_0, \ldots, U_{n+1})$. If $m(U_{n+1}) \geq m_0(U_{n+1}) + \delta$ then

$$\sum_{i=0}^{n+1} m(U_i) \geq \sum_{i=0}^{n+1} = m_0(U_i) - \delta/2^{n+1} + m_0(U_{n+1}) + \delta$$

$$= 1 - [(n + 1)/2^{n+1}] \delta + \delta > 1.$$ 

Clearly this holds for each $U_i$ so $|m(U_i) - m_0(U_i)| < \delta$ for $0 \leq i \leq n + 1$. This gives the inequalities

$$\sum_{i=1}^{n+1} m_0(U_i) a_i \leq \int f dm_0 \leq \sum_{i=0}^{n+1} m_0(U_i) a_i$$

and

$$\sum_{i=1}^{n+1} [m_0(U_i) - \delta] a_i \leq \int f dm \leq \sum_{i=0}^{n+1} [m_0(U_i) + \delta] a_i.$$
Therefore
\[ [m_0(U_0) + \delta](-a_0) - \sum_{i=1}^{n+1} [m_0(U_i)(a_i - a_{i-1}) + \delta a_i] \leq \int \! f \, dm_0 - \int \! f \, dm \]
\[ \leq m_0(U_0)a_0 + \sum_{i=1}^{n+1} [m_0(U_i)(a_i - a_{i-1}) + \delta a_{i-1}]. \]

Now \( a_0 < (\varepsilon/8) \) implies \( m(U_0)a_0 < (\varepsilon/8) \) and \( \delta a_0 < (\varepsilon/8) \). Furthermore,
\[ \sum_{i=1}^{n+1} m_0(U_i)(a_i - a_{i-1}) \leq \frac{1}{n} \sum_{i=1}^{n+1} m(U_i) \leq \frac{1}{n} \leq \frac{\varepsilon}{8}. \]

and
\[ \sum_{i=1}^{n+1} \delta a_{i-1} < \sum_{i=1}^{n+1} \delta a_i = \delta \sum_{i=1}^{n+1} a_i < \delta(n + 1) < \frac{\varepsilon}{8}. \]

So \( |\int \! f \, dm_0 - \int \! f \, dm| < \varepsilon \).

In order to accommodate
\[ N(m_0; \varepsilon; U) \cap N(m_0; \varepsilon; f_j) \]
consider \( N^*(m_0; \delta_1; U_1, \ldots, U_n) \) and \( N^*(m_0; \delta_2; O_1, \ldots, O_k) \). Let \( U_{ij} = U_i \cap O_j \) and \( \delta = (\min\{\delta_1, \delta_2\}/\max\{k, n\}) \). Note that for \( m \in P(X) \), \( m(U_i) \geq \sum_{j=1}^{k} m(U_{ij}) \) and \( m(U_j) \geq \sum_{i=1}^{n} m(U_{ij}) \) and for \( m_0 \) these are equalities.

Suppose \( m \in N^*(m_0; \delta; U_1, \ldots, U_n) \). So \( m(U_{ij}) > m_0(U_{ij}) - \delta \) implies
\[ m(U_i) \geq \sum_{j=1}^{k} m(U_{ij}) > \sum_{j=1}^{k} [m_0(U_{ij}) - \delta] \]
\[ = m_0(U_i) - k\delta > m_0(U_i) - \delta_1. \]

Similarly \( m(O_j) > m_0(O_j) - \delta_2 \), so \( m \) belongs to the intersection of \( N^*(m_0; \delta_1; U_1, \ldots, U_n) \) and \( N^*(m_0; \delta_2; O_1, \ldots, O_k) \).

\( N^*(m_0; \varepsilon; U_1, \ldots, U_n) \) is open since it is equal to
\[ \bigcap_{i=1}^{n} \{m \in P(X): m(U_i) > m_0(U_i) - \varepsilon\} \]
and these sets are open. \( \square \)

The following lemma follows directly from the Baire Category Theorem. It is stated as a lemma in the form that will be used in the example to follow.

**Lemma 2.** Suppose \( X \) is topologically complete, \( V \) is a closed subset of \( X \), \( O \subseteq V \) open relative to \( V \), \( N \subseteq V \) that is first category in \( V \) and \( \langle K_i \rangle \) is a sequence of closed sets in \( X \) such that \( O - N \subseteq \bigcup_{i=1}^{\infty} K_i \). There exists a set \( U \subseteq O \) with the following properties:

(i) \( U \) is open relative to \( O \),
(ii) \( \overline{U} \subseteq O \) (\( \overline{U} \) denoting the closure of \( U \) in \( X \)),
(iii) \( \overline{U} \subseteq K_j \) for some \( j \),
(iv) the diameter of \( \overline{U} < (1/2^i) \) where \( i \) is an arbitrary natural number.
III. Main result. We now introduce the following notation. Let \( X \) be a perfect, complete, separable metric space and \( Y \) a countable dense subset of \( X \). Let \( Y = \{y_1, y_2, \ldots \} \) and \( Y_i = \{y_1, y_2, \ldots, y_i\} \). Let \( i_1, i_2, i_3, \ldots, i_n \) denote an increasing sequence of positive integers, \( C_{ij} = \{m \in P(X) \mid m(y_j) \geq 1 - (1/j)\} \), \( V_n = \bigcap_{i=1}^n C_{i,k} \) and \( C_j = \bigcup_{i=j}^\infty C_{ij} \). Note that \( C_{ij} \) is closed in \( P(X) \) and \( \bigcap_{j=1}^\infty C_j = P(Y) \). Assume that \( P(X) \) has been equipped with a complete metric.

**Lemma 3.** \( V_n \cap C_{i,n+1} \) is nowhere dense in \( V_n \).

**Proof.** \( V_n \cap C_{i,n+1} \) is closed in \( V_n \) so it remains to show that if \( m_0 \in V_n \cap C_{i,n+1} \) then any neighborhood of \( m_0 \) contains a point of \( V_n - C_{i,n+1} \).

Consider \( N^*(m_0; \varepsilon; U_1, \ldots, U_k) \). Since \( X \) is perfect and complete and \( Y \) is countable we may choose \( a_i \in U_j - Y \) for \( 1 \leq j \leq k \).

**Case 1.** \( m_0(Y, ) < 1 - 1/(n + 1) \). Define \( m \in P(X) \) by the following: \( m(y_j) = m_0(y_j) \) for \( y_j \in Y, \) and \( m(a_j) = m_0(U_j) - m(U_j \cap Y, ) \) for \( 1 \leq j \leq k \). Clearly \( m(U_j) = m_0(U_j) \) and \( m \) agrees with \( m_0 \) on \( Y, \) but \( m(Y, ) < 1 - 1/(n + 1) \). So \( m \in N^*(m_0; \varepsilon; U_1, \ldots, U_k) \cap [V_n - C_{i,n+1}] \).

**Case 2.** \( m_0(Y, ) \geq 1 - (1/n + 1) \). Let \( t \) denote the first integer between \( 1 \) and \( i_n \) inclusive with the property that \( \sum_{j=1}^t m_0(y_j) \geq 1 - 1/(n + 1) \). Pick \( 0 < \varepsilon < 1 \) so that

\[
1 - \frac{1}{n + 1} < \sum_{j=1}^t m_0(y_j) + \varepsilon m_0(y_t) < 1 - \frac{1}{n + 1}.
\]

Define \( m \) by the following: \( m(y_j) = m_0(y_j) \) for \( j < t \), \( m(y_t) = \varepsilon m_0(y_t) \) and \( m(a_j) = m_0(U_j) - m(U_j \cap Y_t) \) for \( 1 \leq j \leq k \). Now, \( m(U_j) = m_0(U_j) \) and \( m \) agrees with \( m_0 \) on \( Y_t \). Since \( m_0 \in V_n \), \( m(Y, ) \geq 1 - (1/j) \) for \( i_j < t \). If \( t \leq i_j \leq i_n \) then \( m(Y, ) \geq 1 - 1/(n + 1) \). Hence, \( m \in V_n \). On the other hand, \( m(Y, ) < 1 - 1/(n + 1) \) implies that \( m \not\in C_{i,n+1} \). This completes the proof. \( \Box \)

**Lemma 4.** \( V_n \cap C_{i,n+1} \) is dense in \( V_n \).

**Proof.** Let \( m_0 \in V_n \) and consider \( N^*(m_0; \varepsilon; U_1, U_2, \ldots, U_k) \). Since \( Y \) is dense and \( X \) is perfect, \( (Y, - Y, ) \cap U_j \neq 0 \). Pick \( i > i_n \) with the property that \( (Y, - Y, ) \cap U_j \neq 0 \). Pick \( a_j \in (Y, - Y, ) \cap U_j \). Define \( m \) by the following: \( m(y_j) = m_0(y_j) \) for \( 1 \leq j \leq i_n \) and \( m(a_j) = m_0(U_j) - m(U_j \cap Y, ) \). Clearly \( m(Y, ) = 0 \), so \( m \in C_{i,n+1} \). Furthermore \( m \) agrees with \( m_0 \) on \( Y_n \) implies \( m \in V_n \) and \( m(U_j) = m_0(U_j) \) implies \( m \in N^*(m_0; \varepsilon; U_1, \ldots, U_k) \). \( \Box \)

**Example 5.** In this example the notation introduced before Lemma 3 will be used. It will be shown that \( P(Y) \) is not a \( G_{\delta} \) subset of \( P(X) \). Assume that \( P(X) \) has been equipped with complete metric.

Suppose \( P(Y) \) is a \( G_{\delta} \) subset of \( P(X) \). Then \( P(X) - P(Y) \) is an \( F_{\delta} \) subset of \( P(X) \) and there exist closed sets \( \langle K_{ij} \rangle \) such that \( P(Y)^c = \bigcap_{j=1}^\infty K_j \), where \( K_j = \bigcup_{i=j}^\infty K_{ij} \).

Using Lemmas 2, 3 and 4 a sequence of sets, \( \langle \tilde{O}_n \rangle \), will be constructed that have the property that their intersection belongs to both \( P(Y) \) and \( P(Y)^c \). Recall the notation introduced before Lemma 3.
Let $i_1 = 1$. Then $V_1 = C_{l_1}$ is a closed subset of $P(X)$. By Lemma 3 we have that $V_1 \cap C_2$ is a first category $F_\sigma$ set with respect to $V_1$. Let $O_1 = P(X)$. Now $O_1 - (V_1 \cap C_2) \subseteq C_2 \subseteq P(Y)^c \subseteq K_1$ (note $O_1 \subseteq V_1$ and in general $O_n$ will be a subset of $V_n$). By Lemma 2 there exists a set $U_1 \subseteq O_1$ having properties (i)-(iv) as stated in the lemma. For property (iv) require that the diameter of $\overline{U}_1 < \frac{1}{2}$. Note that $K_1$ corresponds to the union of the $K_n$ of Lemma 2.

Since $U_1$ is open in $O_1$ and, by Lemma 4, $C_2$ is dense in $V_1$, we can pick $i_2 > i_1$ so that $C_{i_2} \cap U_1$. Let $V_2 = V_1 \cap C_{i_2}$ and $O_2 = U_1 \cap V_2$. Note that $O_2$ is open in $V_2$ and $V_2$ is a closed subset of $P(X)$ which is complete. Proceed to the $n + 1$ step.

Assume that $i_1 < \cdots < i_n$ have been defined, $V_n = \bigcap_{k=1}^n C_{i_k}$, $O_n \supseteq O_2 \cdots O_n$ have been defined so that the diameter of $O_k < (1/2^{k-1})$ and $O_k \subseteq V_k$ is open relative to $V_k$. By Lemmas 3 and 4, $V_n \cap C_{n+1}$ is a first category $F_\sigma$ subset of $V_n$ and is dense in $V_n$. $O_n - (V_n \cap C_{n+1}) \subseteq C_{n+1} \subseteq P(Y)^c \subseteq K_n$, so by Lemma 2 there exists $U_n \subseteq O_n$ having properties (i)-(iv) (where the diameter of $\overline{U}_n < (1/2^n)$). Pick $i_{n+1} > i_n$ so that $C_{i_{n+1}} \cap U_n$. Let $V_{n+1} = V_n \cap C_{i_n}, n + 1$ and $O_{n+1} = U_n \cap V_{n+1}$.

The sets $O_1 \supseteq O_2 \supseteq \cdots$, are nonempty and the diameter of $\overline{O}_n < (1/2^{n-1})$ for $n > 1$. Since $P(X)$ is complete there exists $m \in \bigcap_{n=1}^\infty O_n$. For each $n$, $O_n \subseteq V_n$ and $V_n$ is closed so $\overline{O}_n \subseteq V_n$ and $m \in V_n$. Therefore, for each $n, m \in C_{i_n}$, so $m \in C_n$ and $m \in \bigcap_{n=1}^\infty C_n = P(Y)$.

On the other hand $\overline{O}_n \subseteq \overline{U}_{n-1}$ for $n \geq 2$. By property (iii) of Lemma 2, $\overline{U}_{n-1} \subseteq K_{n-1}$, which gives that $m \in K_n$ for each $n$ and $m \in \bigcap_{n=1}^\infty K_n = P(Y)^c$. \hfill $\square$

**Theorem 6.** If $X$ is separable metric and coanalytic but not topologically complete, then $X$ contains a countable, dense in itself, $G_\delta$ subspace.

**Proof.** See [1 or 5].

**Theorem 7.** If $X$ is a complete separable metric space and $Y \subseteq X$ and $P(Y)$ is a $G_{\delta\sigma}$ subset of $P(X)$ then $P(Y)$ is in fact a $G_\delta$ subset of $P(X)$.

**Proof.** Suppose $Y$ fails to be a $G_\delta$ subset of $X$ (and hence, by Theorem B, $Y$ is not topologically complete). $Y$ and $X$ are homeomorphic to the degenerate measures (point masses) in $P(Y)$ and $P(X)$, see [4], so $Y$ is a $G_{\delta\sigma}$ subset of $X$. Since $X$ is complete, $Y$ is coanalytic. By Theorem 6 there exists a $G_\delta$ subset of $X$, call it $G$, such that $G \cap Y$ is countable and dense in itself. Let $Y_1 = G \cap Y$ and $X_1 = \overline{Y}_1 \cap G$, i.e. $X_1$ is the closure of $Y_1$ in $G$. $X_1$ is dense in itself and topologically complete. $Y_1$ is a countable dense subset of $X_1$.

Since $Y_1 = Y \cap X_1$ we have that $P(Y_1) = P(Y) \cap P(X_1)$. By hypothesis $P(Y)$ is a $G_{\delta\sigma}$ in $P(X)$, so $P(Y_1)$ is a $G_{\delta\sigma}$ in $P(X_1)$. On the other hand, $X_1$ and $Y_1$ have the properties specified in Example 5 in which it was shown that $P(Y_1)$ is not a $G_{\delta\sigma}$ in $P(X_1)$; hence, $Y$ must be a $G_\delta$ subset of $X$ and therefore, by Theorems A and B, $P(Y)$ is a $G_\delta$ subset of $P(X)$. \hfill $\square$

**Remark.** It is not difficult to show that if $Y$ belongs to the $\alpha$th multiplicative Borel class with respect to $X$, then $P(Y)$ belongs to the $\alpha$th multiplicative Borel class with respect to $P(X)$. In light of statements (2) and (3) of §1, is the following true?
If $P(Y)$ belongs to the $\beta$th, $\beta > 0$, additive Borel class with respect to $P(X)$, then there exists $\alpha < \beta$ such that $P(Y)$ belongs to the $\alpha$th multiplicative Borel class with respect to $P(Y)$.

References

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