THE CHAIN RECURRENT SET FOR
MAPS OF THE INTERVAL

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ABSTRACT. Let $f$ be a continuous map of a compact interval into itself. We show that if the set of periodic points of $f$ is a closed set then every chain recurrent point is periodic.

1. Introduction. The concept of chain recurrent points and the chain recurrent set was introduced by Conley [3, 4] in the study of flows on manifolds. This paper is concerned with the chain recurrent set of a continuous map of the interval to itself. Our main result is the following.

THEOREM. Let $f$ be a continuous map of a compact interval $I$ into itself. If the set of periodic points of $f$ is a closed set then every chain recurrent point is periodic.

One may think of a chain recurrent point as a point that looks periodic on a computer, i.e. where error is allowed (see §2 for the definition). Thus, the theorem asserts that (if the hypothesis is satisfied) every point that looks periodic on a computer is actually periodic.

Nitecki [8] and Ziong [10] prove that if the hypothesis of our theorem is satisfied then every nonwandering point is periodic. Previous related results and special cases may be found in [1 and 5].

Note that while the nonwandering set is always a subset of the chain recurrent set the reverse inclusion need not hold. For example, consider a map $f$ from $[0, 1]$ to itself such that:

1. $f(0) = 0, f(\frac{1}{2}) = \frac{1}{2}, f(\frac{3}{2}) = 1, \text{ and } f(1) = 0.$
2. $f$ maps the interval $[0, \frac{1}{2}]$ homeomorphically onto itself.
3. $f(x) > x$ for all $x \in (0, \frac{1}{2})$.
4. The restriction of $f$ to each of the intervals $[\frac{1}{2}, \frac{3}{2}]$ and $[\frac{3}{2}, 1]$ is linear.

Then each point in the interval $(0, \frac{1}{2})$ is wandering, but chain recurrent.

Before proving the theorem, we give some definitions and obtain some basic properties of the chain recurrent set which hold for continuous maps of a compact metric space to itself.

2. Chain recurrent points and the chain recurrent set. In this section we let $f$ be a continuous map from a compact metric space $(X, d)$ into itself. Let $x, y \in X$. An $\varepsilon$-chain from $x$ to $y$ is a finite sequence of points \{$x_{0}, x_{1}, \ldots, x_{n}$\} of $X$ with $x = x_{0}$. 

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We say \( x \) can be chained to \( y \) if for every \( \varepsilon > 0 \) there is an \( \varepsilon \)-chain from \( x \) to \( y \), and we say \( x \) is chain recurrent if \( x \) can be chained to itself. The set of all chain recurrent points is called the chain recurrent set and denoted by \( R(f) \).

For any nonnegative integer \( n \) we define \( f^n \), inductively by letting \( f^0 \) be the identity map and \( f^n = f \circ f^{n-1} \). A point \( x \in X \) is called a periodic point of \( f \) if \( f^n(x) = x \) for some positive integer \( n \), and the smallest such \( n \) is called the period of \( x \).

The following property of the chain recurrent set (which may be easily verified) is false for the nonwandering set \([6]\).

**Lemma 1.** \( R(f) = R(f^n) \).

**Lemma 2.** If \( x \in R(f) \) then \( f^k(x) \) can be chained to \( x \) for every positive integer \( k \).

**Proof.** Since the conclusion follows immediately if \( x \) is a periodic point of \( f \), we may assume that \( x \) is not periodic. Let \( \delta_i \) be the distance from \( x \) to the set \( \{ f(x), f^2(x), \ldots, f^{k+1}(x) \} \) and let \( \varepsilon > 0 \). By the uniform continuity of \( f, f^2, \ldots, f^{k+1} \), there is a \( \delta > 0 \) such that if \( \gamma, z \in X \) and \( d(\gamma, z) < \delta \) then \( d(f(\gamma), f(z)) < \min\{\varepsilon/(k+1), \delta_i/(k+1)\} \) for \( i = 0, 1, \ldots, k+1 \). Let \( \{x_0, x_1, \ldots, x_n\} \) be a \( \delta \)-chain from \( x \) to \( x \). Note that for \( i = 1, \ldots, k + 1 \),

\[
d(f'(x), x_i) \leq d(f'(x_0), f^{i-1}(x_1)) + d(f^{i-1}(x_1), f^{i-2}(x_2)) + \cdots + d(f(x_{i-1}), x_i) \leq \delta_i.
\]

Since \( d(f'(x), x) > \delta_1 \), we see that \( x_i \neq x \) for \( i = 1, \ldots, k + 1 \). Hence any \( \delta \)-chain from \( x \) to \( x \) must have more than \( k + 2 \) elements. A similar argument shows that \( d(f^{k+1}(x), x_{k+1}) < \varepsilon \). Hence \( \{f^k(x), x_{k+1}, x_{k+2}, \ldots, x_n\} \) is an \( \varepsilon \)-chain from \( f^k(x) \) to \( x_n = x \). Q.E.D.

We say a set \( Y \subset X \) is positively chain invariant if for every \( y \in Y \) and \( x \in X \setminus Y \), \( y \) cannot be chained to \( x \). The next lemma follows directly from Lemma 2 and this definition.

**Lemma 3.** Let \( Y \) be a positively chain invariant set. If \( x \notin Y \) and \( f^k(x) \in Y \) for some positive integer \( k \) then \( x \notin R(f) \).

3. **Proof of the Theorem.** For the remainder of the paper we let \( I \) denote the interval \([0, 1]\) and \( f \) be a continuous map of \( I \) to itself.

The next lemma gives a sufficient condition for a subinterval of \( I \) to be positively
chain invariant.

**Lemma 4.** Let \([a, b]\) be a subinterval of \([0, 1]\) with \( f([a, b]) \subset [a, b] \) and \( b \neq 1 \).
Suppose there is a neighborhood \( W \) of \( b \) with \( f(W) \subset [a, b] \). Suppose also that either \( f(a) > a \) or \( a = 0 \). Then \([a, b]\) is positively chain invariant.
Proof. Let \( z \in [a, b] \) and \( y \in I \setminus [a, b] \). We must show that \( z \) cannot be chained to \( y \).

Let \( \epsilon_1 \) be a positive number smaller than the distance from \( b \) to the right endpoint of \( W \). If \( a = 0 \) let \( \epsilon_2 = \epsilon_1 \). If \( a \neq 0 \) let \( \epsilon_2 \) be a positive number such that if \( x \in (a - \epsilon_2, a) \) then \( f(x) \in (a, b + \epsilon_1/2) \). By choosing \( \epsilon_1 \) and \( \epsilon_2 \) smaller if necessary we may assume that \( y \notin (a - \epsilon_2, b + \epsilon_1) \). Let \( \epsilon = \min\{\epsilon_2, \epsilon_1/2\} \).

Suppose that \( z \) can be chained to \( y \). Then there is an \( \epsilon \)-chain \( \{x_0, x_1, \ldots, x_n\} \) from \( z \) to \( y \). By choice of \( \epsilon \), it follows that if \( 0 < k < n \) and \( x_k \in (a - \epsilon_2, b + \epsilon_1) \) then \( x_{k+1} \in (a - \epsilon_2, b + \epsilon_1) \).

Hence, each element of the chain \( \{x_0, x_1, \ldots, x_n\} \) is in the interval \( (a - \epsilon_2, b + \epsilon_1) \). This is a contradiction since \( x_n = y \) and \( y \notin (a - \epsilon_2, b + \epsilon_1) \). Q.E.D.

The following lemma is a slight modification of Lemma 6 of [1]. The proof given in [1] applies in this case.

Lemma 5. Let \( W \) be an open interval in \( I \) which contains no periodic points of \( f \). Suppose that if an endpoint of \( W \) is not a fixed point of \( f \) then it is an endpoint of \( I \) and is not periodic.

Suppose also that for some \( y_0 \in W \), \( f(y_0) < y_0 \). Then for every positive integer \( n \) and every \( y \in W \), \( f^n(y) < y \).

As noted in [8], it follows from [2] that if \( f \) has a periodic point whose period is not a power of two, then the set of periodic points of \( f \) is not a closed set. Thus, from [7] (or Lemma 1.6 of [9]) we obtain the following.

Lemma 6. Suppose that the set of periodic points of \( f \) is a closed set, and suppose that for some \( z \in I \) and some positive integer \( m \), \( f(z) < z \leq f^m(z) \). Then there is a point \( s_0 \in I \) such that \( f^i(z) > s_0 \) for \( i = 0, 2, 4, \ldots, m - 2 \), and \( f^i(z) < s_0 \) for \( i = 1, 3, 5, \ldots, m - 1 \).

Proof of the Theorem. Let \( x \in I \) be any point which is not periodic. We will show that \( x \) is not chain recurrent.

Let \( W_1 \) denote the component of the complement (in \( I \)) of the set of periodic points of \( f \) with \( x \in W_1 \). Let \( W \) be the unique open interval such that \( W = W_1 \). Then either \( x \in W \) or \( x \) is an endpoint of \( I \) and an endpoint of \( W \).

By Lemma 1, to show that \( x \) is not chain recurrent for \( f \), we may show that \( x \) is not chain recurrent for \( f^k \). Hence, without loss of generality, we may assume that \( W \) satisfies the hypothesis of Lemma 5, and \( f^n(y) < y \) for every \( y \in W \) and every positive integer \( n \).

Let \( p \) denote the left endpoint of \( W \). Then \( f(p) = p \) and \( x \neq p \).

Let \( K = \bigcup_{n=0}^{\infty} f^n([p, x]) \). Note that \( K \) is an interval whose right endpoint is \( x \) and \( f(K) \subset K \). Let \( a \) denote the left endpoint of \( K \).

We claim that \( x \notin f(K) \). It follows from Lemma 5 and the definition of \( K \) that \( x \notin f(K) \). Thus, to prove the claim, it suffices to show that \( f(a) \neq x \) (since \( f(K) = f(K) \)). If \( a \in K \) it follows that \( f(a) \neq x \), so we may assume that \( a \notin K \). Note that \( a \in f(K) = f(K) \), but \( a \notin f(K) \) (since \( f(K) \subset K \)). Thus, since \( a \) is the only element of \( K \setminus K \), \( f(a) = a \). Hence \( f(a) \neq x \). This establishes our claim that \( x \notin f(K) \).
We will show that \( \overline{f(K)} \) can be extended to a positively chain invariant set which does not contain \( x \) and does contain \( f(x) \). Once this is shown, it follows from Lemma 3 that \( x \) is not chain recurrent.

Note that \( a \) (the left endpoint of \( K \)) is also the left endpoint of \( f(K) \). Let \( c \) denote the right endpoint of \( f(K) \). Then \( a < c < x \). Let \( b \) denote the midpoint of the interval \([c, x]\). Then \( f([a, b]) \subseteq [a, b] \).

If \( f(a) \neq a \) or \( a = 0 \) then by Lemma 4, \( [a, b] \) is positively chain invariant and, by Lemma 3, \( x \) is not chain recurrent.

Thus we may assume that \( f(a) = a \) and \( a = 0 \). Since \( f \) and \( f^2 \) are uniformly continuous, there is a \( \delta_1 > 0 \) such that if \( |y_1 - y_2| < \delta_1 \) then \( |f(y_1) - f(y_2)| < b - a \) and \( |f^2(y_1) - f^2(y_2)| < b - a \). Also, there is a \( \delta_2 > 0 \) with \( \delta_2 < \delta_1 \) such that if \( |y_1 - y_2| < \delta_2 \) then \( |f(y_1) - f(y_2)| < \delta_1 \).

Let \( y \in (a - \delta_2, a) \). We will show that \( f^k(y) < b \) for every positive integer \( k \). Suppose that for some positive integer \( k \), \( f^k(y) \geq b \). Then for some nonnegative integer \( r \) with \( r < k \), \( f^r(y) \in (a - \delta_2, a) \) and \( f^{r+1}(y) \leq a - \delta_2 \). Let \( z = f^r(y) \). Then \( f(z) < z \). Also, if \( m = k - r \) then \( f^m(z) > z \).

By Lemma 6, there is a point \( s_0 \in f \) such that \( f^i(z) > s_0 \) for \( i = 0, 2, 4, \ldots, m - 2 \), and \( f^i(z) < s_0 \) for \( i = 1, 3, 5, \ldots, m - 1 \). Since \( f(z) < s_0 \) and \( z \in (a - \delta_2, a) \), it follows from the choice of \( \delta_1 \) that \( s_0 \in (a - \delta_1, a) \).

Note that the points \( z, f^2(z), f^4(z), \ldots, f^m(z) \) are all to the right of \( s_0 \). None of these points can be in the interval \([a, b] \) since \( f([a, b]) \subseteq [a, b] \). Hence, it follows from the choice of \( \delta_1 \) (and by induction) that each of these points is in the interval \((s_0, a) \). Thus, by choice of \( \delta_1 \), \( f^m(z) < b \). This is a contradiction since \( f^m(z) = f^k(y) \geq b \). Thus, for every \( y \in (a - \delta_2, a) \) and every positive integer \( k \), \( f^k(y) < b \).

Let \( K_1 = \bigcup_{k=0}^{\infty} f^k([-a - \delta_2, a]) \). Here, in case \( a - \delta_2 < 0 \), we make the convention that \( f^k(S) = f^k(S \cap I) \). Then \( K_1 \) is an interval and for all \( y \in K_1 \), \( y < b \). Let \( a_1 \) denote the left endpoint of \( K_1 \). If \( a_1 = 0 \) or \( f(a_1) > a_1 \) then the interval \([a_1, b] \) satisfies the hypothesis of Lemma 4 and thus, \( x \) is not chain recurrent. Hence, we may assume that \( a_1 = 0 \) and \( f(a_1) = a_1 \).

Let \( K_2 = \bigcup_{k=0}^{\infty} f^k([-a_1 - \delta_2, a_1]) \). Note that if \( y \in (a_1 - \delta_2, a_1) \) then \( f^k(y) < b \) for every positive integer \( k \). This follows by the same proof used above to show that if \( y \in (a - \delta_2, a) \) then \( f^k(y) < b \) for every positive integer \( k \). Thus, for all \( y \in K_2 \), \( y < b \). Let \( a_2 \) denote the left endpoint of \( K_2 \). If \( a_2 = 0 \) or \( f(a_2) > a_2 \) then the interval \([a_2, b] \) satisfies the hypothesis of Lemma 4 and \( x \) is not chain recurrent. Hence, we say assume that \( f(a_2) = a_2 \).

We repeat the above argument inductively forming points \( a_n \) with \( a_{n+1} \leq a_n - \delta_2 \). Since the interval \([0, 1]\) has finite length, for some positive integer \( n \) we must have either \( a_n = 0 \) or \( f(a_n) > a_n \). It follows that the interval \([a_n, b] \) satisfies the hypothesis of Lemma 4 and \( x \) is not chain recurrent. Q.E.D.

References


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