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THE CHAIN RECURRENT SET FOR MAPS OF THE INTERVAL

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ABSTRACT. Let f be a continuous map of a compact interval into itself. We show that if the set of periodic points of f is a closed set then every chain recurrent point is periodic.

1. Introduction. The concept of chain recurrent points and the chain recurrent set was introduced by Conley [**3, 4**] in the study of flows on manifolds. This paper is concerned with the chain recurrent set of a continuous map of the interval to itself. Our main result is the following.

THEOREM. *Let f be a continuous map of a compact interval I into itself. If the set of periodic points of f is a closed set then every chain recurrent point is periodic.*

One may think of a chain recurrent point as a point that looks periodic on a computer, i.e. where error is allowed (see §2 for the definition). Thus, the theorem asserts that (if the hypothesis is satisfied) every point that looks periodic on a computer is actually periodic.

Nitecki [**8**] and Ziong [**10**] prove that if the hypothesis of our theorem is satisfied then every nonwandering point is periodic. Previous related results and special cases may be found in [**1** and **5**].

Note that while the nonwandering set is always a subset of the chain recurrent set the reverse inclusion need not hold. For example, consider a map f from $[0, 1]$ to itself such that:

(1) $f(0) = 0, f(\frac{1}{3}) = \frac{1}{3}, f(\frac{2}{3}) = 1$, and $f(1) = 0$.

(2) f maps the interval $[0, \frac{1}{3}]$ homeomorphically onto itself.

(3) $f(x) > x$ for all $x \in (0, \frac{1}{3})$.

(4) The restriction of f to each of the intervals $[\frac{1}{3}, \frac{2}{3}]$ and $[\frac{2}{3}, 1]$ is linear.

Then each point in the interval $(0, \frac{1}{3})$ is wandering, but chain recurrent.

Before proving the theorem, we give some definitions and obtain some basic properties of the chain recurrent set which hold for continuous maps of a compact metric space to itself.

2. Chain recurrent points and the chain recurrent set. In this section we let f be a continuous map from a compact metric space (X, d) into itself. Let $x, y \in X$. An *ε-chain* from x to y is a finite sequence of points $\{x_0, x_1, \dots, x_n\}$ of X with $x = x_0$,

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$y = x_n$ and $d(f(x_{i-1}), x_i) < \varepsilon$ for $i = 1, \ldots, n$. We say x can be chained to y if for every $\varepsilon > 0$ there is an ε-chain from x to y, and we say x is *chain recurrent* if x can be chained to itself. The set of all chain recurrent points is called the chain recurrent set and denoted by $R(f)$.

For any nonnegative integer n we define f^n, inductively by letting f^0 be the identity map and $f^n = f \circ f^{n-1}$. A point $x \in X$ is called a periodic point of f if $f^n(x) = x$ for some positive integer n, and the smallest such n is called the period of x.

The following property of the chain recurrent set (which may be easily verified) is false for the nonwandering set [**6**].

LEMMA 1. $R(f) = R(f^n)$.

LEMMA 2. *If $x \in R(f)$ then $f^k(x)$ can be chained to x for every positive integer k.*

PROOF. Since the conclusion follows immediately if x is a periodic point of f, we may assume that x is not periodic. Let δ_1 be the distance from x to the set $\{f(x), f^2(x), \ldots, f^{k+1}(x)\}$ and let $\varepsilon > 0$. By the uniform continuity of $f, f^2, \ldots, f^{k+1}$, there is a $\delta > 0$ such that if $y, z \in X$ and $d(y, z) < \delta$ then $d(f^i(y), f^i(z)) < \min\{\varepsilon/(k+1), \delta_1/(k+1)\}$ for $i = 0, 1, \ldots, k+1$. Let $\{x_0, x_1, \ldots, x_n\}$ be a δ-chain from x to x. Note that for $i = 1, \ldots, k+1$,

$$\begin{aligned} d(f^i(x), x_i) &\leq d(f^i(x_0), f^{i-1}(x_1)) + d(f^{i-1}(x_1), f^{i-2}(x_2)) \\ &\quad + \cdots + d(f(x_{i-1}), x_i) \\ &= d(f^{i-1}(f(x_0)), f^{i-1}(x_1)) + d(f^{i-2}(f(x_1)), f^{i-2}(x_2)) \\ &\quad + \cdots + d(f(x_{i-1}), x_i) \\ &< \frac{\delta_1}{k+1} + \frac{\delta_1}{k+1} + \cdots + \frac{\delta_1}{k+1} \leq \delta_1. \end{aligned}$$

Since $d(f^i(x), x) > \delta_1$, we see that $x_i \neq x$ for $i = 1, \ldots, k+1$. Hence any δ-chain from x to x must have more than $k+2$ elements. A similar argument shows that $d(f^{k+1}(x), x_{k+1}) < \varepsilon$. Hence $\{f^k(x), x_{k+1}, x_{k+2}, \ldots, x_n\}$ is an ε-chain from $f^k(x)$ to $x_n = x$. Q.E.D.

We say a set $Y \subset X$ is *positively chain invariant* if for every $y \in Y$ and $x \in X \setminus Y$, y cannot be chained to x. The next lemma follows directly from Lemma 2 and this definition.

LEMMA 3. *Let Y be a positively chain invariant set. If $x \notin Y$ and $f^k(x) \in Y$ for some positive integer k then $x \notin R(f)$.*

3. Proof of the Theorem. For the remainder of the paper we let I denote the interval $[0, 1]$ and f be a continuous map of I to itself.

The next lemma gives a sufficient condition for a subinterval of I to be positively chain invariant.

LEMMA 4. *Let $[a, b]$ be a subinterval of $I = [0, 1]$ with $f([a, b]) \subset [a, b]$ and $b \neq 1$. Suppose there is a neighborhood W of b with $f(W) \subset [a, b]$. Suppose also that either $f(a) > a$ or $a = 0$. Then $[a, b]$ is positively chain invariant.*

PROOF. Let $z \in [a, b]$ and $y \in I \setminus [a, b]$. We must show that z cannot be chained to y.

Let ε_1 be a positive number smaller than the distance from b to the right endpoint of W. If $a = 0$ let $\varepsilon_2 = \varepsilon_1$. If $a \neq 0$ let ε_2 be a positive number such that if $x \in (a - \varepsilon_2, a)$ then $f(x) \in (a, b + \varepsilon_1/2)$. By choosing ε_1 and ε_2 smaller if necessary we may assume that $y \notin (a - \varepsilon_2, b + \varepsilon_1)$. Let $\varepsilon = \min\{\varepsilon_2, \varepsilon_1/2\}$.

Suppose that z can be chained to y. Then there is an ε-chain $\{x_0, x_1, \ldots, x_n\}$ from z to y. By choice of ε, it follows that if $0 \leqslant k \leqslant n$ and $x_k \in (a - \varepsilon_2, b + \varepsilon_1)$ then $x_{k+1} \in (a - \varepsilon_2, b + \varepsilon_1)$.

Hence, each element of the chain $\{x_0, x_1, \ldots, x_n\}$ is in the interval $(a - \varepsilon_2, b + \varepsilon_1)$. This is a contradiction since $x_n = y$ and $y \notin (a - \varepsilon_2, b + \varepsilon_1)$. Q.E.D.

The following lemma is a slight modification of Lemma 6 of [**1**]. The proof given in [**1**] applies in this case.

LEMMA 5. *Let W be an open interval in I which contains no periodic points of f. Suppose that if an endpoint of W is not a fixed point of f then it is an endpoint of I and is not periodic.*

Suppose also that for some $y_0 \in W$, $f(y_0) < y_0$. Then for every positive integer n and every $y \in W$, $f^n(y) < y$.

As noted in [**8**], it follows from [**2**] that if f has a periodic point whose period is not a power of two, then the set of periodic points of f is not a closed set. Thus, from [**7**] (or Lemma 1.6 of [**9**]) we obtain the following.

LEMMA 6. *Suppose that the set of periodic points of f is a closed set, and suppose that for some $z \in I$ and some positive integer m, $f(z) < z \leqslant f^m(z)$. Then there is a point $s_0 \in I$ such that $f^i(z) > s_0$ for $i = 0, 2, 4, \ldots, m-2$, and $f^i(z) < s_0$ for $i = 1, 3, 5, \ldots, m-1$.*

PROOF OF THE THEOREM. Let $x \in I$ be any point which is not periodic. We will show that x is not chain recurrent.

Let W_1 denote the component of the complement (in I) of the set of periodic points of f with $x \in W_1$. Let W be the unique open interval such that $\overline{W} = \overline{W_1}$. Then either $x \in W$ or x is an endpoint of I and an endpoint of W.

By Lemma 1, to show that x is not chain recurrent for f, we may show that x is not chain recurrent for f^k. Hence, without loss of generality, we may assume that W satisfies the hypothesis of Lemma 5, and $f^n(y) < y$ for every $y \in W$ and every positive integer n.

Let p denote the left endpoint of W. Then $f(p) = p$ and $x \neq p$.

Let $K = \cup_{n=0}^{\infty} f^n([p, x])$. Note that K is an interval whose right endpoint is x and $f(K) \subset K$. Let a denote the left endpoint of K.

We claim that $x \notin \overline{f(K)}$. It follows from Lemma 5 and the definition of K that $x \notin f(K)$. Thus, to prove the claim, it suffices to show that $f(a) \neq x$ (since $f(\overline{K}) = \overline{f(K)}$. If $a \in K$ it follows that $f(a) \neq x$, so we may assume that $a \notin K$. Note that $a \in \overline{f(K)} = f(\overline{K})$, but $a \notin f(K)$ (since $f(K) \subset K$). Thus, since a is the only element of $\overline{K} \setminus K$, $f(a) = a$. Hence $f(a) \neq x$. This establishes our claim that $x \notin \overline{f(K)}$.

We will show that $\overline{f(K)}$ can be extended to a positively chain invariant set which does not contain x and does contain $f(x)$. Once this is shown, it follows from Lemma 3 that x is not chain recurrent.

Note that a (the left endpoint of K) is also the left endpoint of $f(K)$. Let c denote the right endpoint of $f(K)$. Then $a < c < x$. Let b denote the midpoint of the interval $[c, x]$. Then $f([a, b]) \subset [a, b]$.

If $f(a) \neq a$ or $a = 0$ then by Lemma 4, $[a, b]$ is positively chain invariant and, by Lemma 3, x is not chain recurrent.

Thus we may assume that $f(a) = a$ and $a \neq 0$. Since f and f^2 are uniformly continuous, there is a $\delta_1 > 0$ such that if $|y_1 - y_2| < \delta_1$ then $|f(y_1) - f(y_2)| < b - a$ and $|f^2(y_1) - f^2(y_2)| < b - a$. Also, there is a $\delta_2 > 0$ with $\delta_2 < \delta_1$ such that if $|y_1 - y_2| < \delta_2$ then $|f(y_1) - f(y_2)| < \delta_1$.

Let $y \in (a - \delta_2, a)$. We will show that $f^k(y) < b$ for every positive integer k. Suppose that for some positive integer k, $f^k(y) \geq b$. Then for some nonnegative integer r with $r < k$, $f^r(y) \in (a - \delta_2, a)$ and $f^{r+1}(y) \leq a - \delta_2$. Let $z = f^r(y)$. Then $f(z) < z$. Also, if $m = k - r$ then $f^m(z) > z$.

By Lemma 6, there is a point $s_0 \in I$ such that $f^i(z) > s_0$ for $i = 0, 2, 4, \ldots, m - 2$, and $f^i(z) < s_0$ for $i = 1, 3, 5, \ldots, m - 1$. Since $f(z) < s_0$ and $z \in (a - \delta_2, a)$, it follows from the choice of δ_2 that $s_0 \in (a - \delta_1, a)$.

Note that the points $z, f^2(z), f^4(z), \ldots, f^{m-2}(z)$ are all to the right of s_0. None of these points can be in the interval $[a, b]$ since $f([a, b]) \subset [a, b]$. Hence, it follows from the choice of δ_1 (and by induction) that each of these points is in the interval (s_0, a). Thus, by choice of δ_1, $f^m(z) < b$. This is a contradiction since $f^m(z) = f^k(y) \geq b$. Thus, for every $y \in (a - \delta_2, a)$ and every positive integer k, $f^k(y) < b$.

Let $K_1 = \bigcup_{k=0}^{\infty} f^k([a - \delta_2, a])$. Here, in case $a - \delta_2 < 0$, we make the convention that $f^k(S) = f^k(S \cap I)$. Then K_1 is an interval and for all $y \in K_1$, $y \leq b$. Let a_1 denote the left endpoint of K_1. If $a_1 = 0$ or $f(a_1) > a_1$ then the interval $[a_1, b]$ satisfies the hypothesis of Lemma 4 and thus, x is not chain recurrent. Hence, we may assume that $a_1 \neq 0$ and $f(a_1) = a_1$.

Let $K_2 = \bigcup_{k=0}^{\infty} f^k([a_1 - \delta_2, a_1])$. Note that if $y \in (a_1 - \delta_2, a_1)$ then $f^k(y) < b$ for every positive integer k. This follows by the same proof used above to show that if $y \in (a - \delta_2, a)$ then $f^k(y) < b$ for every positive integer k. Thus, for all $y \in K_2$, $y \leq b$. Let a_2 denote the left endpoint of K_2. If $a_2 = 0$ or $f(a_2) > a_2$ then the interval $[a_2, b]$ satisfies the hypothesis of Lemma 4 and x is not chain recurrent. Hence, we say assume that $f(a_2) = a_2$.

We repeat the above argument inductively forming points a_n with $a_{n+1} \leq a_n - \delta_2$. Since the interval $[0, 1]$ has finite length, for some positive integer n we must have either $a_n = 0$ or $f(a_n) > a_n$. It follows that the interval $[a_n, b]$ satisfies the hypothesis of Lemma 4 and x is not chain recurrent. Q.E.D.

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