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A HOPF MODULE CHARACTERIZATION OF HOPF ALGEBRAS

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ABSTRACT. A bialgebra over a field is a Hopf algebra if and only if all (nonzero) right Hopf modules are free, as modules, on a set of invariants.

0. Introduction. Let A be a Hopf algebra over a field k. The fact that any (nonzero) right A-Hopf module M is free as an A-module on any linear basis of the invariants of M is of profound importance in the theory of Hopf algebras [S, Theorem 4.1.1]. In this paper we prove a converse—we show that Hopf algebras over k are those bialgebras over k whose (nonzero) right Hopf modules are free as a module on a set of invariants. Thus the converse characterizes Hopf algebras over a field in terms of Hopf modules.

Actually we show that for a bialgebra A over k to be a Hopf algebra, it is sufficient to know that the (nonzero) finitely generated right A-Hopf modules, which are free as an A-module, have a module basis of invariants. Thus what distinguishes bialgebras from Hopf algebras is not so much a matter of Hopf modules being free but rather a matter of free Hopf modules having a certain type of basis.

1. Preliminaries. Let C be a coalgebra over a field k, and suppose (M, ω) is a finite-dimensional right C-comodule. Let $m_1, \ldots, m_s$ be a basis for M and let $e_{ij} \in C$ for $1 \leqslant i, j \leqslant s$ be defined by $\omega(m_j) = \Sigma_i m_i \otimes e_{ij} \in M \otimes_k C$. Then the equations

$$(1.1) \qquad \Delta e_{ij} = \sum_k e_{ik} \otimes e_{kj},$$

and

$$(1.2) \qquad \varepsilon(e_{ij}) = \delta_{ij} \quad \text{for } 1 \leqslant i, j \leqslant s$$

follow from the equations $\omega \otimes I \circ \omega = I \otimes \Delta \circ \omega$ and $I \otimes \varepsilon \circ \omega = I$ respectively.

Now suppose A is a bialgebra over k and (M, ω) is any right A-comodule. Write $\omega(m) = \Sigma \underline{m}_{(1)} \otimes m_{(2)} \in M \otimes_k A$ for $m \in M$. Then $(M \otimes_k A, \omega_A)$ is a right A-comodule where $\omega_A(m \otimes a) = \Sigma(\underline{m}_{(1)} \otimes a_{(1)}) \otimes m_{(2)}a_{(2)}$ for $m \in M$ and $a \in A$. Regarding $M \otimes_k A$ as a right A-module by extension of scalars $((m \otimes a) \circ b = m \otimes ab)$ it is easy to check that $M \otimes_k A$ is a right A-Hopf module. Note that $M \otimes_k A$ is free as an A-module if $M \neq (0)$.

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For any right A-comodule M we say $m \in M$ is an *invariant* if $\omega(m) = m \otimes 1$ and let $M_f \subseteq M$ denote the subspace of invariants of M. The purpose of this section is to prove the following.

LEMMA. *Let A be a bialgebra over a field k, and suppose (M, ω) is a finite-dimensional right A-comodule. Assume further $(M \otimes_k A, \omega_A)$ has a module basis of invariants. Then if $m_1, \ldots, m_s$ is a linear basis for M, the $s \times s$ matrix $(e_{ij}) \in M_s(A)$ is invertible, where e_{ij} is defined by $\omega(m_j) = \Sigma_i m_i \otimes e_{ij}$.*

PROOF. Let $x = \Sigma_{ji} m_j \otimes a_j \in M \otimes_k A$. Then by definition

$$\omega_A(x) = \sum_{j,k} (m_k \otimes a_{j(1)}) \otimes e_{kj} a_{j(2)},$$

so x is invariant if and only if $\Sigma_j a_{j(1)} \otimes e_{kj} a_{j(2)} = a_k \otimes 1$ for $1 \leqslant k \leqslant s$.

Note that $m_1 \otimes 1, \ldots, m_s \otimes 1$ is a module basis for $M \otimes_k A$. Since $\ker \varepsilon$ is a cofinite ideal of A, any module basis has s elements. Let $x_1, \ldots, x_s \in (M \otimes_k A)_f$ be a module basis for $M \otimes_k A$, and write $x_i = \Sigma_j m_j \otimes a_{ji}$ where $a_{ji} \in A$. Since x_i is invariant it follows $\Sigma_j a_{ji(1)} \otimes e_{kj} a_{ji(2)} = a_{ki} \otimes 1$ for $1 \leqslant k \leqslant s$. Applying $\varepsilon \otimes I$ to this equation we deduce

$$\sum_j e_{kj} a_{ji} = \varepsilon(a_{ki})1 \quad \text{for } 1 \leqslant i, k \leqslant s. \tag{1.3}$$

Observe $x_j = \Sigma_i (m_i \otimes 1) a_{ij}$ means that the $s \times s$ matrix $(a_{ij}) \in M_s(A)$ is a change of basis matrix, hence is invertible. Therefore $(\varepsilon(a_{ij})1)$ is invertible, so by 1.3 (e_{ij}) is the product of invertible matrices. Q.E.D.

2. The main theorem. We characterize those bialgebras over a field which are Hopf algebras.

THEOREM. *Let A be a bialgebra over a field k. Then the following are equivalent*:

(a) *Any (nonzero) right A-Hopf module is free, as a module, and any linear basis of invariants is a module basis.*

(b) *Any (nonzero) finitely generated right A-Hopf module is free, as a module, and has a module basis consisting of invariants.*

(c) *Any (nonzero) right A-Hopf module, which is finitely generated and free as an A-module, has a module basis consisting of invariants.*

(d) *A is a Hopf algebra.*

PROOF. (d) $\Rightarrow$ (a) is [**S**, Theorem 4.1.1]. (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) are trivial. Thus it suffices to show (c) $\Rightarrow$ (d).

Assume the hypothesis of (c), and let $C \subseteq A$ be a finite dimensional subcoalgebra. To show that A is a Hopf algebra, by [**R**, Lemma 2(b)] it suffices to show that the inclusion $\iota\colon C \to A$ has an inverse in the convolution algebra $\operatorname{Hom}_k(C, A)$, that is there exists a linear map $s\colon C \to A$ satisfying

$$\begin{aligned} s * \iota(c) &= \sum s(c_{(1)}) c_{(2)} = \varepsilon(c)1 \\ &= \sum c_{(1)} s(c_{(2)}) = \iota * s(c) \quad \text{for } c \in C, \end{aligned}$$

Regard $M = C$ as a right A-comodule, and choose a linear basis $m_1, \ldots, m_s$ for M satisfying $\varepsilon(m_i) = \delta_{1i}$ for $1 \leqslant i \leqslant s$. Let $e_{ij} \in A$ be defined by $\Delta m_j = \sum_i m_i \otimes e_{ij}$. Observe that $e_{ij} \in C_{1i}$ since M is in fact a coalgebra, and that $m_j = \sum_i \varepsilon(m_i) e_{1j} = e_{1j}$ for $1 \leqslant j \leqslant s$.

By the lemma $(e_{ij}) \in M_s(A)$ has an inverse (a_{ij}). Define a linear map $s\colon C \to A$ by $s(e_{1j}) = a_{1j}$ for $1 \leqslant j \leqslant s$. Then the computation

$$s * \iota(e_{1i}) = \sum_j a_{1j} e_{ji} = \delta_{1i} 1 = \varepsilon(e_{1i}) 1$$

shows that $s * \iota = \eta \circ \varepsilon$, or equivalently $(s(e_{ij})) \in M_s(A)$ is a left inverse of (e_{ij}). Hence $(s(e_{ij})) = (a_{ij})$, so $(s(e_{ij}))$ is a right inverse of (e_{ij}), or equivalently $\iota * s = \eta \circ \varepsilon$. Q.E.D.

As one may guess, "right" may be replaced by "left" in the Theorem. To see this observe that left A-Hopf modules are right A^{op}-Hopf modules. A^{op} is the bialgebra obtained from A by twisting multiplication and comultiplication, specifically $m^{op} = m \circ T$ and $\Delta^{op} = T \circ \Delta$ where $T\colon A \otimes_k A \to A \otimes_k A (a \otimes b \to b \otimes a)$ is the twist map.

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