

THE GROUP OF AUTOMORPHISMS OF A DISTRIBUTIVELY GENERATED NEAR RING

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ABSTRACT. S. D. Scott has shown that the group of automorphisms of the near ring generated by the automorphisms of a given group is isomorphic to the automorphism group of the given group if that group's automorphism is complete. Here that theorem is generalized by showing that the group of automorphisms of a near ring distributively generated by its units is a subgroup of the group of automorphisms of the group of units. The results obtained are used to find the automorphism groups of certain near rings.

1. Preliminaries. In this paper all near rings considered will be distributively generated (d.g.) and will contain an identity element. Also, they will be written as left-distributive near rings. If K represents an algebraic system, then $\text{Aut}(K)$ ($\text{Inn}(K)$) will denote the group of automorphisms (inner automorphisms) of K . If V is a group, then $A(V)$ will denote the d.g. near ring generated by the automorphisms of V .

The following definitions are as given in [10]. For a near ring N , $D(N)$ designates the (multiplicative) group of distributive units of N . If μ is in $D(N)$, then an inner automorphism of N is defined by taking α to $\mu^{-1}\alpha\mu$ for all α in N . The center of N , denoted by $Z(N)$, is the set of all μ in $D(N)$ such that $\mu^{-1}\alpha\mu = \alpha$ for all α in N . N is said to be complete if $\text{Inn}(N) = \text{Aut}(N)$ and $Z(N)$ is trivial.

Although it was not noted in [10], it is easy to show that if N is d.g. with S as the generating set and if μ is in the center of S , then μ is in $Z(N)$ so that, in fact, $Z(N) = Z(S)$.

If H is a subset of an additive group, then $-H = \{-h \mid h \text{ in } H\}$.

For near ring terminology not defined here or in [10], see [8].

2. Results. In Theorem 4 of [10] S. D. Scott gives an interesting statement about certain near ring automorphisms. However, behind Scott's theorem, there is a more general result. This generalization is incorporated in the next proposition.

PROPOSITION 1. *Let N be a near ring distributively generated by the set D of its distributive units. Then $\text{Aut}(N)$ is embedded in $\text{Aut}(D)$. In particular, if V is an arbitrary group such that $\text{Aut}(V) = D(A(V))$, then $\text{Aut}(A(V)) < \text{Aut}(\text{Aut}(V))$. If β is an automorphism of $(N, +)$ and if β , restricted to (D, \cdot) is an automorphism of (D, \cdot) , then β is an automorphism of N .*

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PROOF. If γ is in $\text{Aut}(N)$, then γ restricted to (D, \cdot) is an automorphism since D contains all distributive units of N . This gives the first assertion. If, for an arbitrary group V , N is taken as $A(V)$ and $\text{Aut}(V) = D(A(V))$, then the second assertion follows from the first.

Let β be as described in the proposition and let n_1, n_2 be in N . Then $n_1 = \sum_{i=1}^r f_i$, $n_2 = \sum_{j=1}^s g_j$, with each f_i and g_j in $-D \cup D$. Hence,

$$(n_1 n_2)\beta = \left(\sum_{j=1}^s \left(\sum_{i=1}^r f_i \right) g_j \right) \beta = \left(\sum_{j=1}^s \left((\pm) \sum_{i=1}^r f_i ((\pm) g_j) \right) \right) \beta$$

("+" holds if g_j in D and "-" holds if g_j in $-D$, i.e. "+" holds if g_j is right-distributive and "-" holds if g_j is antiright distributive)

$$\begin{aligned} &= \left(\sum_{j=1}^s \left((\pm) \sum_{i=1}^r ((f_i \beta) (((\pm) g_j) \beta)) \right) \right) \\ &= \sum_{j=1}^s \left(\left(\sum_{i=1}^r f_i \beta \right) (g_j \beta) \right) = \left(\sum_{i=1}^r (f_i \beta) \right) \left(\sum_{j=1}^s (g_j \beta) \right) = (n_1 \beta)(n_2 \beta). \end{aligned}$$

Thus, β is an automorphism of (N, \cdot) and of N .

COROLLARY 2 (THEOREM 4 OF [10]). *If a group V contains an element v such that $vA(V) = V$ and if $\text{Aut}(V)$ is a complete group, then $\text{Aut}(A(V)) \cong \text{Aut}(V)$ and $A(V)$ is complete.*

PROOF. Since $\text{Aut}(V)$ is complete, $Z(\text{Aut}(V))$ is trivial. Hence,

$$\text{Aut}(V) \cong \text{Aut}(V)/Z(\text{Aut}(V)) \cong \text{Inn}(\text{Aut}(V)) = \text{Aut}(\text{Aut}(V))$$

so that $\text{Aut}(V) \cong \text{Aut}(\text{Aut}(V))$. Since there exists a v in V such that $vA(V) = V$, it follows that $D(A(V)) = \text{Aut}(V)$. The near ring $A(V)$ is, of course, d.g. with $\text{Aut}(V)$ as the generating set. Then it follows from the comment in §1 that $Z(\text{Aut}(V)) = Z(A(V))$ so that $Z(A(V))$ is trivial. Now, it follows that

$$\text{Inn}(A(V)) \cong D(A(V))/Z(A(V)) \cong \text{Aut}(V)/Z(\text{Aut}(V)) \cong \text{Aut}(V).$$

The first isomorphism above is given by Proposition 1 of [10]. Then

$$\text{Aut}(V) \cong \text{Inn}(A(V)) \triangleleft \text{Aut}(A(V)) \leq \text{Aut}(\text{Aut}(V)) \cong \text{Aut}(V).$$

(The normality statement follows from Proposition 2 of [10] and the other inclusion comes from Proposition 1 of this paper.) From this statement we conclude that $\text{Aut}(A(V)) \cong \text{Aut}(V)$. If $\text{Aut}(V)$ is finite, the last equation is sufficient to show that $\text{Inn}(A(V)) = \text{Aut}(A(V))$ and establish that $A(V)$ is complete. If $\text{Aut}(V)$ is infinite, the proof can be given as in [10] where use is made of the fact that $\text{Inn}(A(V))$, as a complete group, is a direct summand of $\text{Aut}(A(V))$.

Scott's proof of his Theorem 4 is more elegant than the proof given here. However, the proof as given here may be more helpful for seeing connections among the groups discussed. Also, the applications given in §3 deal with finite groups.

It is of interest to note the roles played by the two conditions included in the hypothesis of Corollary 2. The completeness condition makes $\text{Aut}(\text{Aut}(V)) \cong \text{Aut}(V)$ and so makes Proposition 1 easier to apply. The role of the other condition is discussed below.

As noted in the proof of the Corollary, the existence of a v in V such that $vA(V) = V$ is used to show that $D(A(V))$ contains no elements other than those of $\text{Aut}(V)$. That, in general, $D(A(V))$ may be larger than $\text{Aut}(V)$ can be seen by looking at any of the groups of odd order described in [2 or 4]. Each of these groups G is a nonabelian group such that $A(G)$ is a ring. Thus any unit in $A(G)$ is distributive. In particular, the identity map added to itself produces a unit which is not in $\text{Aut}(G)$.

A relation between the existence of such a v and the structure of $A(V)$ is given in Proposition 4. But first we need the following result of G. Berman and R. J. Silverman [1].

PROPOSITION 3. *Let e be an idempotent in the near ring N . Then each n in N has a unique decomposition in the form $n = (n - en) + en$. Thus $N = B_e + M_e$, where $B_e = \{n - en \mid n \in N\} = \{t \in N \mid et = 0\}$ and $M_e = \{en \mid n \in N\}$. Also, $B_e \cap M_e = \{0\}$.*

It was noted in [5] that B_e is a right ideal of N and M_e is a subnear ring.

PROPOSITION 4. *Let V be a group. If $A(V)$ contains an idempotent element e such that $M_e(A(V)) \cong V$, then V contains an element v such that $vA(V) = V$.*

PROOF. Assume that $A(V)$ contains an idempotent e such that $M_e(A(V)) \cong V$. Since the sum $A(V) = B_e + M_e$ is semidirect, the group $A(V)/B_e$ is isomorphic to M_e . Since, for any α in $A(V)$, $(\alpha - e\alpha) - e(\alpha - e\alpha) = \alpha - e\alpha$, B_e is modular. By Proposition 2.1 of [6] this all implies that V is a cyclic $A(V)$ -module. However, saying that V is a cyclic $A(V)$ -module is just another way of saying that there exists a v with the desired property.

The converse of Proposition 4 does not hold. Let Q_n designate the generalized quaternion group of order 2^n , $n > 3$. (See [3] for a discussion of the facts about Q_n which are cited below). In Q_n an element of order 4 not contained in the cyclic subgroup of order 2^{n-1} can be mapped under an appropriate automorphism to any element of order 4 not in the cyclic subgroup of order 2^{n-1} . From this it follows that Q_n is a cyclic $A(Q_n)$ -module. However, $A(Q_n)$ contains no idempotents except the trivial ones: the zero endomorphism and the identity automorphism. Thus there is no idempotent e in $A(Q_n)$ such that $M_e(A(Q_n))$ is isomorphism to Q_n .

3. Applications. In this section uses of Scott's theorem and Proposition 1 are given. The automorphism group of $A(D_{2n})$ is found for certain n , where D_{2n} is the dihedral group of order $2n$. We note (see p. 170 of [7]) that $\text{Aut}(D_{2n})$ is isomorphic to the holomorph of the cyclic group of order n .

PROPOSITION 5. *If n is odd, $A(D_{2n})$ is complete and $\text{Aut}(A(D_{2n})) \cong \text{Hol}(C_n)$.*

PROOF. By Schenkman [9, p. 96], $\text{Hol}(C_n)$ is complete if n is odd. Thus $\text{Aut}(D_{2n})$ is complete. Also, for any two elements of D_{2n} of order 2 which are not contained in the cyclic subgroup of order n , there exists an automorphism of D_{2n} which maps the one element to the other. Since these elements of order 2 constitute a generating set for D_{2n} , D_{2n} is a cyclic $A(D_{2n})$ -module with any of these elements of order 2 able to serve as the generator. The proposition now follows directly from Scott's theorem.

The situation with respect to even values of n seems to be more complicated. We study the situation for $n = 6$.

PROPOSITION 6. *All automorphisms of $A(D_{12})$ are inner. Also, $\text{Inn}(A(D_{12})) \cong D_6$.*

PROOF. Let D_{12} be given as $\langle a, b \mid a^6 = e, b^2 = e, ab = ba^5 \rangle$. Automorphisms of D_{12} will be given in the form $[c, d]$ where c is the image of a and d is the image of b . From Table 12/3 of [11] we have that $D_{12} \cong \text{Aut}(D_{12}) \cong \text{Aut}(\text{Aut}(D_{12}))$. From Proposition 1 we have that $\text{Aut}(A(D_{12})) \leq \text{Aut}(\text{Aut}(D_{12}))$. From Proposition 1 of [10],

$$\text{Inn}(A(D_{12})) \cong D(A(D_{12}))/Z(A(D_{12})).$$

But D_{12} is a cyclic $A(D_{12})$ -module so that $D(A(D_{12})) = \text{Aut}(D_{12}) \cong D_{12}$. Since D_{12} has a center of order 2, we have, by use of the comment in §1, that $\text{Inn}(A(D_{12})) \cong D_6$. Thus, since $\text{Aut}(\text{Aut}(D_{12}))$ has order 12, either each automorphism of $\text{Aut}(D_{12})$ generates an automorphism of $A(D_{12})$ or the only automorphisms of $A(D_{12})$ are the inner automorphisms.

From Table 12/3 of [11], $\alpha = [a, ab]$ and $\beta = [a^5, b]$ are generators of $\text{Aut}(D_{12})$. Hence, an automorphism of $\text{Aut}(D_{12})$ can be described in terms of its action on these generators. It is a matter of easy computation to show that $[\alpha, \beta]$, $[\alpha, \alpha^4\beta]$, $[\alpha, \alpha^2\beta]$, $[\alpha^5, \beta]$, $[\alpha^5, \alpha^2\beta]$, and $[\alpha^5, \alpha^4\beta]$ are the inner automorphisms of $\text{Aut}(D_{12})$. Then $[\alpha, \alpha\beta]$ in $\text{Aut}(\text{Aut}(D_{12}))$ is not inner. Since $\text{Aut}(D_{12})$ additively generates $A(D_{12})$, Proposition 1 tells us that $[\alpha, \alpha\beta]$ is in $\text{Aut}(A(D_{12}))$ if and only if it can be extended to an automorphism of $(A(D_{12}), +)$.

An element of $A(D_{12})$ can be presented by giving, in turn, the images of $e, a, a^2, \dots, a^5; b, ab, a^2b, a^3b, a^4b, a^5b$. Thus α can be written as $(e a a^2 \dots a^5; ab a^2b a^3b a^4b a^5b b)$ and

$$\alpha^5\beta = (e a^5 a^4 \dots a; ab b a^5b a^4b a^3b a^2b).$$

Then it follows that $\alpha + \alpha^5\beta = (e e \dots e; e a^2 a^4 e a^2 a^4)$ which is an element of order 3 in $(A(D_{12}), +)$. Now consider $(\alpha + \alpha^5\beta)[\alpha, \alpha\beta]$, which must be given as $(\alpha)[\alpha, \alpha\beta] + (\alpha^5\beta)[\alpha, \alpha\beta]$. But the latter expression is equal to $\alpha + \beta$. Since

$$\beta = (e a^5 a^4 \dots a; b a^5b a^4b a^3b a^2b ab),$$

$\alpha + \beta = (e e \dots e; a a^3 a^5 a a^3 a^5)$, which has order 6. Thus $[\alpha, \alpha\beta]$ does not extend to an automorphism of $A(D_{12})$ and $\text{Aut}(A(D_{12})) = \text{Inn}(A(D_{12}))$.

By a similar argument the following proposition may be established.

PROPOSITION 7. *All automorphisms of $A(D_8)$ are inner. Also, $\text{Inn}(A(D_8))$ is isomorphic to the Klein group.*

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